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Study of control systems under quadratic nonholonomic constraints. Motion planning, introduction to the regularised continuation method.

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THESIS

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Laboratoire de Mathématiques de l'INSA
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THÈSE

Étude des systèmes de contrôle sous contraintes nonholonomes quadratiques. Planification de trajectoires, introduction à la méthode de continuation régularisée.

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Titre

Étude des systèmes de contrôle sous contraintes nonholonomes quadratiques. Planification de trajectoires, introduction à la méthode de continuation régularisée.

Mots-Clés: Système non-linéaire de contrôle, Contraintes non holonomes quadratiques, Équivalence par bouclage, Formes normales, Algèbre de Lie des symétries infinitésimales, Planification de trajectoires, Méthode de continuation, Régularisation de Tikhonov

Résumé

Dans cette thèse, nous nous intéressons à la théorie et aux applications du contrôle géométrique.

Le contrôle géométrique remonte aux travaux de Hermann et Brockett dans les années soixante, a pris son essor dans les années soixante-dix grâce à Hermes, Jurdjevič, Sussmann et a continué dans les années quatre-vingt avec Agrachev, Bonnard, Isidori, Jakubczyk, Nijmeijer, Respondek, Sontag, van der Schaft, et bien d'autres. Cette recherche s'est poursuivie et a établi la théorie du contrôle géométrique comme un domaine de recherche fructueux à la croisée du contrôle non-linéaire, de la géométrie et des équations différentielles. Elle a donné lieu à des monographies par Nijmeijer et van der Schaft [Nv90], Isidori [Isi95], Jurjevič [Jur96], Bullo et Lewis [BL05], Agrachev et Sachkov [AS13], Bloch [Blo15]. La première partie du manuscrit traite de la géométrie différentielle et de la description de certaines orbites des systèmes de contrôles sous l'action des transformations par bouclage. La seconde partie se concentre sur les applications et nous développons un algorithme pour le problème de planification de trajectoire.

La première partie de cette thèse est dédiée au problème d'équivalence des sous-variétés du fibré tangent. Nous considérons $\mathcal{X}$ une variété lisse de dimension n, équipée de coordonnées locales x ; une sous-variété $\mathcal{S}$ du fibré tangent $T\mathcal{X}$ est donnée par une équation de la forme $S(x\,\dot{x}) = 0$ et décrit une équation différentielle implicite (sous-déterminée) du premier ordre. Nous disons que deux telles sous-variétés sont équivalentes si elles sont équivalentes par une transformation ponctuelle (un difféomorphisme) à multiplication par une fonction scalaire non-nulle près. Nous sommes particulièrement intéressés par une caractérisation et une classification des sous-variétés quadratique, c'est-à-dire les sous-variétés données par l'ensemble de niveau zéro d'une équation de la forme

$$S_q(x, \dot{x}) = \dot{x}^t \mathsf{g}(x)\dot{x} + 2\omega(x)\dot{x} + h(x).$$

Les sous-variétés quadratiques décrivent dans chaque espace tangent une quadrique (au sens de la géométrie affine classique). Nous montrons que le problème d'équivalence des sous-variétés peut être étudié sous le prisme de la transformation par bouclage des systèmes de contrôles. Précisément, à une sous-variété $\mathcal{S} \subset T\mathcal{X}$ nous attachons deux systèmes de contrôles qui jouent le rôle d'une représentation paramétrique de $\mathcal{S}$. Le premier est non-linéaire par rapport aux contrôles, et le second est affine ; nous les appelons, respectivement, première et seconde prolongation

de $\mathcal{S}$. Nous montrons que l'équivalence de deux sous-variétés peut être interprétée comme l'équivalence (par le bouclage) de leurs premières et secondes prolongations respectives. Par conséquent, en utilisant la machinerie de la théorie du contrôle géométrique, nous construisons une théorie des systèmes de contrôle quadratiques, c'est-à-dire des systèmes de contrôles décrivant une équation différentielle implicite quadratique.

Quand la variété sous-jacente $\mathcal{X}$ est de dimension 2, i.e. c'est une surface, nous donnons une caractérisation des sous-variétés coniques, qui inclue les formes régulières que sont les sous-variétés elliptiques, hyperboliques, et paraboliques, mais aussi un passage lisse d'un type vers un autre. Nous identifions une classe de système de contrôle affine (sur une variété de dimension 3, avec un contrôle scalaire) qui décrivent la paramétrisation d'une sous-variété conique. Nous proposons une caractérisation de cette classe de système de contrôle affine, qui par l'équivalence de nos problèmes produit une caractérisation des sous-variétés coniques. Les conditions que nous établissons impliquent des fonctions de structure bien définies et attachées à n'importe quel système de contrôle affine. En analysant nos conditions, nos donnons une forme normale des sous-variétés coniques. Ensuite, nous nous intéressons au problème de classification des sous-variétés coniques régulières : elliptiques, hyperboliques, et paraboliques. Nous étudions ce problème à l'aide de la classification (par le bouclage) de leurs premières prolongations données, respectivement, par

$$\begin{aligned} \Xi_E &: \dot{x} = A(x)\cos(w) + B(x)\sin(w) + C(x), \\ \Xi_H &: \dot{x} = A(x)\cosh(w) + B(x)\sinh(w) + C(x), \\ \Xi_P &: \dot{x} = A(x)w^2 + B(x)w + C(x), \end{aligned}$$

et que nous interprétons comme des systèmes de contrôle non-linéaire (w joue le rôle d'un contrôle scalaire). Par définition A et B sont des vecteurs indépendants, donc ils forment un repère mobile du fibré tangent. Premièrement, nous donnons des conditions, dans le cas elliptiques et hyperboliques, qui garantissent que (A, B) peut être transformer en un repère commutatif et nous prouvons que cela peut toujours être fait dans le cas parabolique. Deuxièmement, dans tout les cas, nous caractérisons les formes où le champ C est constant.

Dans le cas d'une vérité lisse $\mathcal{X}$ de dimension $n \geq 3$ nous donnons une caractérisation des sous-variétés quadratiques paraboloïdes $\mathcal{S}_Q$, c'est-à-dire celles qui sont données par l'ensemble de niveau zéro d'une application de la forme

$$S_Q(x, \dot{x}) = \dot{z} - \dot{y}^t Q(x)\dot{y} - b(x)\dot{y} - c(x),$$

où $x = (z, y)$, avec $y = (y_1, \ldots, y_{n-1})$. Cette classe de sous-variétés généralise la classe des sous-variétés paraboliques étudiée dans le cas $n = 2$. Notre construction est basée sur l'étude géométrique et algébrique d'objets attachées à la première et seconde prolongation de $\mathcal{S}_Q$, qui sont respectivement des systèmes de contrôle non-linéaire et affine de la forme

$$\begin{aligned} \Xi_{p,q} &: \dot{x} = A(x)w^t \mathtt{I}_{p,q} w + B(x)w + C(x), \\ \Sigma_{p,q} &: \dot{x} = A(x)w^t \mathtt{I}_{p,q} w + B(x)w + C(x), \quad \dot{w} = u, \end{aligned}$$

où $\mathtt{I}_{p,q} = \left(\begin{smallmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{smallmatrix}\right)$, et $(A, B_1, \ldots, B_m, C)$ sont des champs de vecteurs lisses. Notre caractérisation est explicite, au sens ou elle peut être testé sur n'importe quel système de contrôle affine au moyen de relations algébriques et différentielles entre des fonctions de structure bien définies attachées au système. A chaque fois que cela est possible, nous donnons une interprétation de nos conditions, soit en donnant leur contre-partie géométrique ou en construisant des formes normales. Ensuite, nous traitons le problème de classification des sous-variétés paraboloïdes $\mathcal{S}_Q$ en proposant une classification de leur première prolongation $\Xi_{pq,}$. Nous explorons les formes normales suivantes

$$\Xi'_{p,q} \,:\, \dot{x} = w^t \mathtt{I}_{p,q} w \frac{\partial}{\partial z} + \sum_{i=1}^{m} w_i \frac{\partial}{\partial y_i} + C(x),$$

$$\Xi'_{p,q} \,:\, \dot{x} = w^t \mathtt{I}_{p,q} w \frac{\partial}{\partial z} + \sum_{i=1}^{m} w_i \frac{\partial}{\partial y_i} + c_0(x) \frac{\partial}{\partial z},$$

$$\Xi'_{p,q} \,:\, \dot{x} = w^t \mathtt{I}_{p,q} w \frac{\partial}{\partial z} + \sum_{i=1}^{m} w_i \frac{\partial}{\partial y_i} + c_0 \frac{\partial}{\partial z}, \quad c_0 \in \mathbb{R}.$$

La première forme correspond à l'existence d'une repère commutatif $A, B_1, \ldots, B_m$ (ce qui correspond à la normalisation $Q = \mathtt{I}_{p,q}$ pour $\mathcal{S}_Q$). Dans la seconde forme, nous caractérisons la sous-classe où le champ C est colinéaire avec A (ce qui est intéressant car cela correspond à S_Q, avec $Q = \mathtt{I}_{p,q}$ et, additionnellement, $b = 0$), et finalement nous donnons des conditions pour que C soit constant (ce qui correspond à $Q = \mathtt{I}_{p,q}$, $b = 0$, et $c \in \mathbb{R}$ pour $\mathcal{S}_Q$). Ce dernier cas conduit à une forme canonique avec $c_0 = \pm 1$ ou $c_0 = 0$. Notre classification des systèmes de contrôle paraboloïde donne de manière équivalente une classification des sous-variétés paraboloïdes.

Nous proposons également une approche pour caractériser directement la classe des sous-variétés paraboloïdes $\mathcal{S}_Q$ sans paramètres, nous les appelons des formes-nulles et elles sont données par l'ensemble de niveau zéro de l'application $S^0_Q = \dot{z} - \dot{y}^t \mathtt{I}_{p,q} \dot{y}$. Cette caractérisation est basée sur l'étude des symétries de leur seconde prolongation donnée par

$$\Sigma^0_{p,q} \,:\, \begin{cases} \dot{z} &= w^t \mathtt{I}_{p,q} w \\ \dot{y} &= w \\ \dot{w} &= u \end{cases} .$$

Nous montrons que l'algèbre de Lie des symétries infinitésimales de $\Sigma^0_{p,q}$ décrit cette classe de système de contrôle affine.

Dans la deuxième partie du manuscrit, nous étudions le problème de planification de trajectoire, c'est-à-dire le problème de la conception et de l'étude d'algorithmes qui calculent des contrôles réalisant une certaine trajectoire cible. Nous considérons un système de contrôle affine de la forme

$$\Sigma \,:\, \dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} u_i(t) g_i(x(t)), \quad x \in \mathbb{R}^n, \quad \text{et} \quad u_i \in L^2([0,T], \mathbb{R}).$$

Nous supposons que l'état initial $x(0) = x_0$ est fixé. Une notion importante de la théorie du contrôle est l'application entrée-sortie, qui à un contrôle $u(t)$ associe le point terminal $x_u(T)$ en temps $T > 0$ de la trajectoire associée

$$E \,:\, L^2([0,T], \mathbb{R}^m) \longrightarrow \mathbb{R}^n$$
$$u \longmapsto x_u(T).$$

L'étude de cette application et particulièrement de ces singularités (c'est à dire des points u où la différentielle $\mathrm{d}E(u)$ n'est pas surjective) est le sujet de beaucoup de travaux ; par exemple Lee et Markus [LM67], Bonnard et Chyba [BC03]. Du point de vue des applications, il y a deux questions très naturelles. La première concerne la description de l'ensemble des points atteignables depuis x_0 en temps T et en utilisant des contrôles prescrits. Et, deuxièmement, étant donnée un point cible $x^\star$, atteignable depuis x_0 en temps T, nous cherchons un contrôle (non-nécessairement unique) $u^\star$ réalisant $E(u^\star) = x^\star$. Ce problème est appelée le problème de planification des trajectoires dans la littérature et nous lui dédions la seconde partie de la thèse.

Nous proposons une régularisation de la méthode de continuation introduite par Chitour et Sussmann [CS98 ; Chi06] au début du siècle. L'idée de la méthode est de commencer avec un contrôle arbitraire u^0 qui, en général, donne un état final x^0 différent de $x^\star$. Puis, nous construisons un chemin π dans $\mathbb{R}^n$ qui relie ce premier essai x^0 avec le point cible $x^\star$. Le cœur de la procédure est la construction d'un relèvement de π dans $L^2([0,T],\mathbb{R}^m)$. A savoir, nous cherchons un chemin $\Pi(s)$ de contrôles satisfaisant $\Pi(0) = u^0$ et pour tout $s \in [0,1]$ nous avons $E(\Pi(s)) = \pi(s)$. Si cette procédure peut être menée jusque $s = 1$ alors $\Pi(1)$ est une solution du problème de planification de trajectoire. La résolution de cette méthode est faite en différenciant par rapport à s la dernière relation, ce qui donne l'équation différentielle

$$\mathrm{d}E(\Pi(s))\Pi'(s) = \pi'(s), \quad \Pi(0) = u^0,$$

appelée l'équation de relèvement de chemin (ERC). Si $\mathrm{d}E(\Pi(s))$ est surjective pour tout s, alors elle possède un inverse à droite, par exemple le pseudo-inverse de Moore-Penrose, ainsi l'ERC a une solution locale. Sous cette hypothèse, nous devons montrer que l'ERC possède une solution globale sur $[0,1]$. Par conséquent, les deux difficultés de la méthode de continuation sont : premièrement, nous devons garantir que $\pi(s)$ évite l'ensemble singulier de E (sinon, à un tel point, l'ERC est n'est pas bien posée), et deuxièmement, il faut que l'ERC ait une solution globale sur $[0,1]$. Actuellement, la description générale des singularités de E est toujours un problème ouvert, et le second point requière l'analyse d'une équation différentielle hautement non-linéaire posée sur un espace de dimension infinie. Donc, il y a une condition globale et une locale pour assurer la faisabilité de la méthode de continuation et la convergence de l'algorithme.

L'idée présentée dans la thèse est d'introduire une régularisation dans la méthode de continuation, celle-ci est inspirée par la régularisation de Tikhonov dans la théorie du pseudo-inverse de Moore-Penrose. Nous proposons une déformation paramétrée de l'inverse à droite de $\mathrm{d}E\Pi(s))$, ce qui donne l'équation

$$\Pi_\lambda'(s) = \mathrm{d}E\left(\Pi_\lambda(s)\right)^* \left(\mathrm{d}E\left(\Pi_\lambda(s)\right)\mathrm{d}E\left(\Pi_\lambda(s)\right)^* + \lambda\,\mathrm{Id}\right)^{-1}\pi'(s), \quad \Pi_\lambda(0) = u^0,$$

appelée l'équation de relèvement de chemin régularisée (ERC-R). Nous montrons que notre méthode résout les deux problèmes de la méthode de continuation classique. En effet, la régularisation assure que l'ERC-R est bien posée pour tout $s \in [0,1]$ et qu'elle admet une solution globale. Dans la thèse nous montrons que la condition pour que la solution Π_λ de l'ERC-R converge ($\lambda \to 0$) vers une solution de l'ERC, est que le chemin $\pi(s)$ satisfait $\pi'(s) \in \operatorname{im}\mathrm{d}E(\Pi(s))$, pour tout s. Finalement, nous illustrons le potentiel de la notre méthode à travers plusieurs exemples numériques.

Title
Study of control systems under quadratic nonholonomic constraints. Motion planning, introduction to the regularised continuation method.

Keywords : Nonlinear control system, Quadratic nonholonomic constraints, Feedback equivalence, Normal forms, Lie algebra of infinitesimal symmetries, Motion planning, Continuation method, Tikhonov regularisation

Abstract

In this thesis, we are interested in theoretical and applied geometric control theory.

Geometric control theory dates back to the work of Hermann and Brockett in the sixties, took off in the seventies due to Hermes, Jurdjevič, Sussmann and followed in the eighties by Agrachev, Bonnard, Isidori, Jakubczyk, Nijmeijer, Respondek, Sontag, van der Schaft, and many others. That research has been continuing and has established geometric control theory as a fruitful research domain at the crossover of nonlinear control, geometry, and differential equations and has led to monographs by Nijmeijer and van der Schaft [Nv90], Isidori [Isi95], Jurjevič [Jur96], Bullo and Lewis [BL05], Agrachev and Sachkov [AS13], Bloch [Blo15].

The first part of the manuscript deals with differential geometry and the description of some orbits of control systems under the action of the group of feedback transformations. The second part is focused on applications and we develop an algorithm for the motion planning problem.

The first part of this thesis is dedicated to the problem of equivalence of submanifolds of a tangent bundle. Consider a smooth n-dimensional manifold $\mathcal{X}$, equipped with local coordinates x; a submanifold $\mathcal{S}$ of the tangent bundle $T\mathcal{X}$ is given by an equation of the form $S(x, \dot{x}) = 0$ and describes an implicit (underdetermined) first order differential equation. We call two such submanifolds equivalent if they are equivalent via a point transformation up to multiplication by a nonvanishing function. We are especially interested in characterising and classifying quadric submanifolds given by the zero level-set of an equation of the form

$$S_q(x, \dot{x}) = \dot{x}^t \mathsf{g}(x)\dot{x} + 2\omega(x)\dot{x} + h(x).$$

Quadric submanifolds describe in each tangent space a quadric (in the sense of classical affine geometry). We show that the problem of equivalence of submanifolds can be studied under the prism of feedback transformations of control systems. Precisely, to a submanifold $\mathcal{S} \subset T\mathcal{X}$ we attach two control systems that plays the role of a parametric representation of $\mathcal{S}$. The first one is control-nonlinear and the second is control-affine; we call them a first and a second prolongation of $\mathcal{S}$. We show that the equivalence (under point transformations, and multiplication by a nonvanishing function) of two submanifolds can be interpreted as the equivalence (via feedback transformations) of their corresponding first and second prolongations. Therefore, using the machinery of geometric control theory, we construct a theory of quadric control systems, that is of control systems describing a quadric implicit first order ordinary differential equation.

When the base manifold $\mathcal{X}$ is 2-dimensional, i.e. it is a surface, we give a characterisation of conic submanifolds, that includes the regular forms of ellipses, hyperbolas, and parabolas, but also a smooth passage from one type to another. We identify a class of control-affine systems (on a 3-dimensional manifold and with scalar control) that describes the parametrisation of conic submanifolds. We propose a characterisation of that class of control-affine system, which by equivalence of our problems provides a characterisation of conic submanifolds. The conditions that we state involve well-defined structure functions attached to any control-affine system. By analysing our conditions, we give a normal form of conic submanifolds. Next, we are interested in the problem of classifying regular conic submanifolds: elliptic, hyperbolic, and parabolic. We study this problem with the help of the classification (via feedback transformations) of their first prolongation given, respectively, by

$$\begin{aligned}
\Xi_E &: \dot{x} = A(x)\cos(w) + B(x)\sin(w) + C(x),\\
\Xi_H &: \dot{x} = A(x)\cosh(w) + B(x)\sinh(w) + C(x),\\
\Xi_P &: \dot{x} = A(x)w^2 + B(x)w + C(x),
\end{aligned}$$

and interpreted as control-nonlinear systems (w plays the role of the scalar control). By definition, A and B are linearly independent vector fields, so they form a moving frame of the tangent bundle. We first give conditions, in the elliptic and hyperbolic case, that guarantee that (A, B) can be transformed into a commutative frame and we prove that this can always be done in the parabolic case. Then, in all cases, we characterise the forms where the vector field C is constant.

In the case of a smooth manifold $\mathcal{X}$ of dimension $n \geq 3$, we give a characterisation of paraboloid submanifolds $\mathcal{S}_Q$, i.e. those that are given by a map of the form

$$S_Q(x, \dot{x}) = \dot{z} - \dot{y}^t Q(x)\dot{y} - b(x)\dot{y} - c(x),$$

where $x = (z, y)$, with $y = (y_1, \ldots, y_{n-1})$. That class of submanifolds generalises the class of parabolic submanifolds studied in the case $n = 2$. Our construction is based on the study of geometric and algebraic objects attached to first and second prolongations of $\mathcal{S}_Q$, which are, respectively, control-nonlinear and control-affine systems of the form

$$\begin{aligned}
\Xi_{p,q} &: \dot{x} = A(x)w^t \mathtt{I}_{p,q} w + B(x)w + C(x),\\
\Sigma_{p,q} &: \dot{x} = A(x)w^t \mathtt{I}_{p,q} w + B(x)w + C(x), \quad \dot{w} = u,
\end{aligned}$$

where $\mathtt{I}_{p,q} = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}$, and $(A, B_1, \ldots, B_m, C)$ are smooth vector fields. Our characterisation is explicit, i.e. it can be tested explicitly on any control-affine system by means of algebraic and differential relations between well-defined structure functions attached to the system. Every time when it is possible we give an interpretation of our conditions either by giving their geometric counterpart or by constructing normal forms. Next, we deal with the problem of classifying paraboloid submanifolds $\mathcal{S}_Q$ by proposing a classification of their first prolongation $\Xi_{p,q}$. We explore the following normal forms,

$$\Xi'_{p,q} : \dot{x} = w^t \mathtt{I}_{p,q} w \frac{\partial}{\partial z} + \sum_{i=1}^{m} w_i \frac{\partial}{\partial y_i} + C(x),$$
$$\Xi'_{p,q} : \dot{x} = w^t \mathtt{I}_{p,q} w \frac{\partial}{\partial z} + \sum_{i=1}^{m} w_i \frac{\partial}{\partial y_i} + c_0(x)\frac{\partial}{\partial z},$$
$$\Xi'_{p,q} : \dot{x} = w^t \mathtt{I}_{p,q} w \frac{\partial}{\partial z} + \sum_{i=1}^{m} w_i \frac{\partial}{\partial y_i} + c_0\frac{\partial}{\partial z}, \quad c_0 \in \mathbb{R}.$$

The first form corresponds to the existence of a commutative frame $A, B_1, \dots, B_m$ (which corresponds to the normalisation $Q = \mathtt{I}_{p,q}$ in $\mathcal{S}_Q$). In the second normal form, we characterise the form where the vector field C is collinear with A (which is interesting because it corresponds to $\mathcal{S}_Q$ with $Q = \mathtt{I}_{p,q}$ and, additionally, $b = 0$), and finally we give conditions for C being constant (corresponding to $Q = \mathtt{I}_{p,q}$, $b = 0$, and $c \in \mathbb{R}$ for $\mathcal{S}_Q$). The latter case leads to a canonical form with $c_0 = \pm 1$ or $c_0 = 0$. Our classification of paraboloid system gives, equivalently, a classification of paraboloid submanifolds.

We also propose an approach to directly characterise paraboloid submanifolds $\mathcal{S}_Q$ with no parameters, called null-forms and given by $S^0_Q = \dot{z} - \dot{y}^t \mathtt{I}_{p,q} \dot{y}$. That characterisation is based on studying symmetries of their second prolongation given by

$$\Sigma^0_{p,q} : \begin{cases} \dot{z} &= w^t \mathtt{I}_{p,q} w \\ \dot{y} &= w \\ \dot{w} &= u \end{cases} .$$

We show that the Lie algebra of infinitesimal symmetries of $\Sigma^0_{p,q}$ describes that class of control-affine systems.

In the second part of the manuscript we study the motion planning problem, that is, the problem of designing and studying algorithms that compute controls realising a certain target trajectory. We consider a control-affine system of the form

$$\Sigma : \dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} u_i(t) g_i(x(t)), \quad x \in \mathbb{R}^n, \quad \text{and} \quad u_i \in L^2([0,T], \mathbb{R}).$$

We assume that a state $x(0) = x_0$ is fixed. An important notion of control theory is the endpoint mapping, which to a control $u(t)$ associates the terminal point $x_u(T)$ in time $T > 0$ of the associated trajectory:

$$\begin{aligned} E : L^2([0,T], \mathbb{R}^m) &\longrightarrow \mathbb{R}^n \\ u &\longmapsto x_u(T). \end{aligned}$$

The study of that map and, particularly, of its singularities (i.e. the points u, where the differential $\mathrm{d}E(u)$ is not surjective) is the subject of many works, e.g. Lee and Markus [LM67], Bonnard and Chyba [BC03]. From the point of view of applications, there are two very natural questions. The first one consists in describing the set of reachable points from x_0 in time T using prescribed controls. And, second, given a target point $x^\star \in \mathbb{R}^n$, reachable from x_0 in time T, we look for a (non necessarily unique) control $u^\star$ realising $E(u^\star) = x^\star$. This problem is called the motion planning problem in the literature and we dedicate the second part of the thesis to it.

We propose a regularisation of the *continuation method* introduced by Chitour and Sussmann [CS98; Chi06] at the beginning of the century. The idea of the method is to start with an arbitrary control u^0 which, in general, gives a final state x^0 different from $x^\star$. Then, we construct a path π in $\mathbb{R}^n$ joining the first guess x^0 to the target point $x^\star$, that is $\pi(0) = x^0$ and $\pi(1) = x^\star$. The core of the procedure is the construction of a lift of π in $L^2([0,T],\mathbb{R}^m)$. Namely, we look for a path $\Pi(s)$ of controls satisfying $\Pi(0) = u^0$ and for all $s \in [0,1]$ we have $E(\Pi(s)) = \pi(s)$. If that procedure can be carried out to $s = 1$ then $\Pi(1)$ is a solution of the motion planning problem. The resolution of this method is done by differentiating with respect to s the last relation, which yield the differential equation

$$\mathrm{d}E(\Pi(s))\Pi'(s) = \pi'(s), \quad \Pi(0) = u^0,$$

called the path lifting equation (PLE). If $\mathrm{d}E(\Pi(s))$ is surjective for all s, then it possesses a right inverse, for instance the Moore-Penrose pseudo-inverse, and thus the PLE has a local solution. Under that assumption, we need to show that the PLE possesses a global solution on $[0,1]$. Therefore, the two difficulties of the continuation method are: first, one needs that $\pi(s)$ avoids the singular set of E (otherwise, at such a point, the PLE is not well-posed), and secondly, one needs to guarantee that the PLE admits a global solution on $[0,1]$. Nowadays, the description of the singular set of E is still an open problem in its generality, and the second point requires the analysis of a highly nonlinear differential equation posed on an infinite-dimensional space. Therefore, there is a global and a local condition to ensure the feasibility of the continuation method and the convergence of the algorithm.

The idea presented in this thesis is to introduce a regularisation, inspired by the Tikhonov regularisation in the theory of the Moore-Penrose pseudo-inverse, in the continuation method. We propose a parameter deformation of the right inverse of $\mathrm{d}E\Pi(s))$, which yields equation

$$\Pi'_\lambda(s) = \mathrm{d}E\left(\Pi_\lambda(s)\right)^* \left(\mathrm{d}E\left(\Pi_\lambda(s)\right)\mathrm{d}E\left(\Pi_\lambda(s)\right)^* + \lambda\,\mathrm{Id}\right)^{-1} \pi'(s), \quad \Pi_\lambda(0) = u^0,$$

called the regularised path lifting equation (R-PLE). We show that our method fixes both issues of the classical continuation method. Indeed, the regularisation ensures that the R-PLE is well-posed for every $s \in [0,1]$ and that it admits a global solution. In the thesis, we show that the condition for a solution Π_λ of the R-PLE to converge $(\lambda \to 0)$ towards a solution of the PLE is that $\pi(s)$ satisfies $\pi'(s) \in \operatorname{im}\mathrm{d}E(\Pi(s))$, for all s. Finally, we illustrate the potential of our method through several numerical examples.

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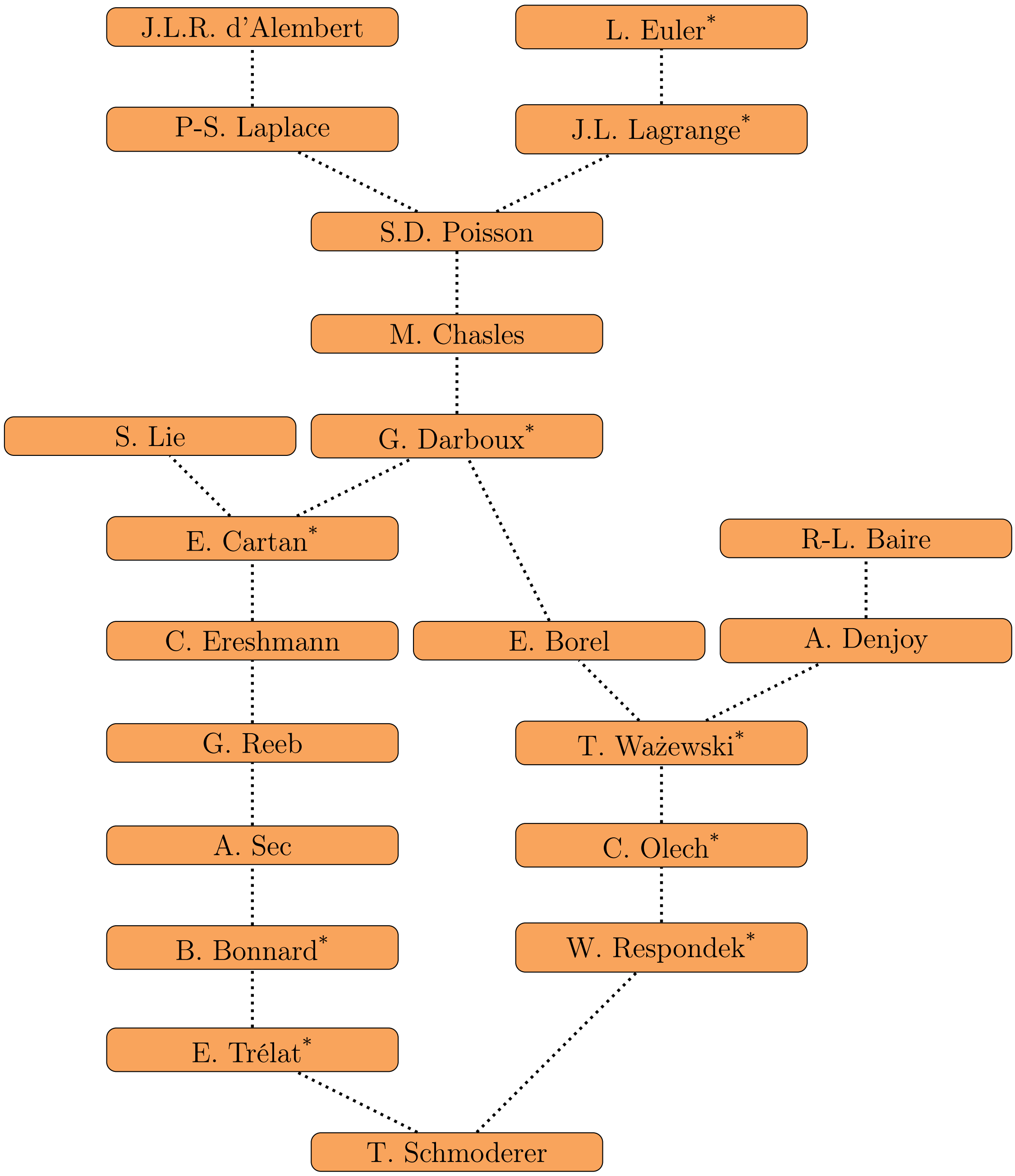


Figure 1: Source: https://www.mathgenealogy.org/index.php

À Manon

Contents

Part I

Characterisation and classification of control systems under quadratic nonholonomic constraints

Chapter 1

Introduction

The first part of this manuscript is devoted to the problem of equivalence of submanifolds of the tangent bundle of a smooth manifold. We are particularly interested in a characterisation and a classification of submanifolds that describe quadric hypersurfaces in the fibers of the tangent bundle. Our analysis relies on attaching to each submanifold two control systems (one fully nonlinear with respect to the controls, and the other control-affine) and then on studying the orbits of those systems under the action of feedback transformations. We will show that the problem of equivalence of submanifolds can equivalently be considered as a problem of equivalence of control systems. Therefore, using the machinery of geometric control we will develop a theory of quadric submanifolds and of their control systems counterparts.

This introduction is organised as follows. In section 1, we define the different mathematical tools that we use. Then, in section 2, we introduce the problem of equivalence of submanifolds of a tangent bundle and develop a passage from the problem of characterising submanifolds to the problem of characterising control systems. Finally, in section 3, we outline the organisation of this part of the manuscript and we summarise the main contributions of it.

1 Preliminaries

In this section, we give the main definitions and notations used in this first part of the manuscript. Throughout this part, the word *smooth* will always mean C^∞-smooth.

1.1 Differential geometry

In this subsection, we introduce some mathematical tools from differential geometry that will be useful for the description of control systems. The reader used to differential geometry may skip this subsection. Allover the manuscript, manifolds are «smooth, finite-dimensional, Hausdorff, second countable, and paracompact»; see [Lee13; Car92] for a detailed introduction. All objects (vector fields, tensor fields, functions) are also smooth. We will consider (if not stated otherwise) embedded submanifolds. For a manifold $\mathcal{X}$ we will denote by $T\mathcal{X}$ and $T^*\mathcal{X}$ the tangent and cotangent bundle, respectively. The space of all smooth vector fields (smooth sections of $T\mathcal{X}$) will be denoted $V^\infty(\mathcal{X})$ and the one of all smooth differential p-forms by $\Lambda^p(\mathcal{X})$, except for smooth functions (0-forms) whose space is denoted $C^\infty(\mathcal{X})$.

Vector fields and tensors calculus. For a diffeomorphism $\phi : \mathcal{X} \to \tilde{\mathcal{X}}$, a vector field $f \in V^\infty(\mathcal{X})$, and a differential p-form $\omega \in \Lambda^p(\tilde{\mathcal{X}})$, we denote by $\phi_* f \in V^\infty(\tilde{\mathcal{X}})$ the push-forward of f, and by $\phi^*\omega \in \Lambda^p(\mathcal{X})$ the pull-back of ω. The (local) flow of a vector field $f \in V^\infty(\mathcal{X})$ is denoted by γ_t^f (for any t such that it is defined). For a differential 1-form ω and a vector field f we denote by $\langle \omega, f \rangle \in \mathbb{R}$ their duality bracket. The Lie derivative of a differential p-form ω along a vector field f will be denoted by $\mathrm{L}_f(\omega)$. In particular, for a function $\lambda \in C^\infty(\mathcal{X})$ and its differential $\mathrm{d}\lambda$ (an exact 1-form) we have

$$\mathrm{L}_f(\lambda) = \langle \mathrm{d}\lambda, f \rangle \quad \text{and} \quad \mathrm{L}_f(\mathrm{d}\lambda) = \mathrm{d}\mathrm{L}_f(\lambda).$$

For any smooth functions α, λ, and μ, the Lie derivative possesses the following properties: $\mathrm{L}_{\alpha f}(\lambda) = \alpha \mathrm{L}_f(\lambda)$, and $\mathrm{L}_f(\lambda\mu) = \mathrm{L}_f(\lambda)\mu + \lambda \mathrm{L}_f(\mu)$. Iterative Lie derivatives are defined by $\mathrm{L}_f^k(\lambda) = \mathrm{L}_f\left(\mathrm{L}_f^{k-1}(\lambda)\right)$, for any $k \geq 2$. For any two vector fields $f, g \in V^\infty(\mathcal{X})$, we define their Lie bracket as a new vector field, denoted $[f, g] \in V^\infty(\mathcal{X})$, such that for any smooth function λ we have

$$\mathrm{L}_{[f,g]}(\lambda) = \mathrm{L}_f(\mathrm{L}_g(\lambda)) - \mathrm{L}_g(\mathrm{L}_f(\lambda)).$$

The Lie bracket possesses the following properties: it is bilinear over $\mathbb{R}$, it is skew-commutative, i.e. $[f, g] = -[g, f]$, and it satisfies the Jacobi identity:

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0, \quad \forall\, f, g, h \in V^\infty(\mathcal{X}).$$

Observe that with this bracket the space $V^\infty(\mathcal{X})$ is thus a Lie algebra. Moreover, for any smooth function α, and any vector fields f, g, and h, we have

$$[f, \alpha g + h] = \alpha [f, g] + \mathrm{L}_f(\alpha)\, g + [f, h].$$

Two vector fields f and g satisfying $[f, g] = 0$ are said to be commuting; since under diffeomorphisms $\phi : \mathcal{X} \to \tilde{\mathcal{X}}$ the Lie bracket is transformed by $[\phi_* f, \phi_* g] = \phi_* [f, g]$, the commutativity property does not depend on coordinates. The celebrated *Flow-box* theorem (also called the «Straightening-out Theorem» or the «Local Linearisation Lemma») asserts that on a given n-dimensional manifold $\mathcal{X}$ there exists a local coordinate system $(x_1, \ldots, x_n)$ such that $f = \frac{\partial}{\partial x_1}$ in a neighbourhood of a point where $f \neq 0$. This can simultaneously be done for a family of (locally) independent vector fields $(f_1, \ldots, f_m)$ if and only if they are mutually commuting. The iterated Lie bracket is denoted by $\mathrm{ad}_f^k g = \left[f, \mathrm{ad}_f^{k-1} g\right]$ for $k \geq 1$ with the convention $\mathrm{ad}_f^0 g = g$; see [Isi95, chapter 1] for a detailed introduction and a proof of those properties.

Remark (Definitions in local coordinates). Let $x = (x_1, \ldots, x_n)$ be a local coordinate system on $\mathcal{X}$ (a smooth n-dimensional manifold). We denote by $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$ the natural frame of $T\mathcal{X}$ and thus any vector field $f \in V^\infty(\mathcal{X})$ admits a local representation of the form

$$f(x) = \sum_{i=1}^{n} f^i(x) \frac{\partial}{\partial x_i},$$

for some smooth functions $f^i(x)$. For a (local) diffeomorphism $\phi : \mathcal{X} \to \tilde{\mathcal{X}}$, we set $\tilde{x} = \phi(x)$ a local coordinate system on $\tilde{\mathcal{X}}$. Consider two vector fields $f, g \in V^\infty(\mathcal{X})$

and a function $\lambda \in C^\infty(\mathcal{X})$, then the previous definitions can be expressed in local coordinates by

$$\begin{aligned}(\phi_* f)(\tilde{x}) &= \frac{\partial \phi}{\partial x}(\phi^{-1}(\tilde{x}))\, f(\phi^{-1}(\tilde{x})), \\ \mathrm{L}_f\,(\lambda)\,(x) &= \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i}(x) f^i(x), \\ \text{and} \quad [f, g]\,(x) &= \frac{\partial g}{\partial x}(x) f(x) - \frac{\partial f}{\partial x}(x) g(x).\end{aligned}$$

◆

Distributions. A distribution $\mathcal{D}$ is map that assigns to each point $x \in \mathcal{X}$ a linear subspace $\mathcal{D}(x)$ of the tangent space $T_x\mathcal{X}$. We say that $\mathcal{D}$ is of constant rank m if $\dim \mathcal{D}(x) = m$ for all $x \in \mathcal{X}$, which is always the case if $\mathcal{D}$ is a subbundle. In this part of the manuscript all distributions are assumed to be of constant rank. Thus, locally, $\mathcal{D}$ is spanned by m independent vector fields $g_1, \ldots, g_m$, which will be denoted by $\mathcal{D} = \operatorname{span}\{g_1, \ldots, g_m\}$. In some occasions it will be useful to consider dual objects, called *codistributions*, that are defined in the following way. The codistribution $\operatorname{ann}(\mathcal{D})$ of a smooth distribution $\mathcal{D}$ is a map that assigns to each $x \in \mathcal{X}$ the linear subspace of $T_x^*\mathcal{X}$ given by the set of covectors that annihilate all vectors in $\mathcal{D}(x)$:

$$\operatorname{ann}(\mathcal{D})\,(x) = \{\varpi \in T_x^*\mathcal{X},\ \forall\, v \in \mathcal{D}(x),\ \langle \varpi, v \rangle = 0\}\,.$$

If $\mathcal{D}$ is of constant rank m, then it is possible to find locally $n-m$ smooth differential 1-forms $\omega_{m+1}, \ldots, \omega_n$ such that $\operatorname{ann}(\mathcal{D}) = \operatorname{span}\{\omega_{m+1}, \ldots, \omega_n\} \subset T^*\mathcal{X}$.

We say that a distribution $\mathcal{D}$ is involutive if it is closed under the Lie bracket, i.e. for any two vector fields $g, g' \in \mathcal{D}$ we have $[g, g'] \in \mathcal{D}$. If, locally, we can find $n-m$ smooth functions $\lambda_{m+1}, \ldots, \lambda_n$ such that $\operatorname{ann}(\mathcal{D}) = \operatorname{span}\{\mathrm{d}\lambda_{m+1}, \ldots, \mathrm{d}\lambda_n\}$, then we say that $\mathcal{D}$ is integrable. The following result relates integrability and involutivity.

Theorem 1.1 (Frobenius)**.** *Consider $\mathcal{X}$ a smooth n-dimensional manifold and let $\mathcal{D} \subset T\mathcal{X}$ be a smooth distribution of constant rank $m \leq n$. The following statements are equivalent:*

(i) *$\mathcal{D}$ is integrable;*

(ii) *$\mathcal{D}$ is involutive;*

(iii) *Locally, there exists a coordinate system $(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)$ of $\mathcal{X}$ in which $\mathcal{D}$ takes the form*

$$\mathcal{D} = \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right\};$$

(iv) *Locally, there exists $n-m$ functions $x_{m+1}, \ldots, x_n$ such that*

$$\operatorname{ann}(\mathcal{D}) = \operatorname{span}\{\mathrm{d}x_{m+1}, \ldots, \mathrm{d}x_n\}\,.$$

See [Lun92] for a proof. A distribution that can be expressed as in statement *(iii)* is said to be rectifiable; in appendix A, we generalise the previous theorem by showing when several integrable distributions can simultaneously be rectified.

Statement *(iii)* of Theorem 1.1 can be interpreted as follows. Given a distribution $\mathcal{D} = \operatorname{span}\{g_1, \dots, g_m\}$, with $g_i \in V^\infty(\mathcal{X})$, we consider an action of diffeomorphisms $\phi : \mathcal{X} \to \mathcal{X}$ and of matrix-valued functions $\beta = \left(\beta^i_j\right) : \mathcal{X} \to GL_m(\mathbb{R})$ on the m-tuples of generators $g = (g_1, \dots, g_m)$ by

$$g_j \longmapsto \tilde{g}_j := \phi_* \left(\sum_{i=1}^m g_i \beta^i_j \right)$$

or, using the vector notation $\tilde{g} = \phi_*(g\beta)$, where $\tilde{g} = (\tilde{g}_1, \dots, \tilde{g}_m)$. Observe that rectifying $\mathcal{D}$ means applying a pair (ϕ, β) such that $\tilde{g}_j = \frac{\partial}{\partial \tilde{x}_j}$, for $1 \le j \le m$, where $\tilde{x} = \phi(x)$. Since $[\tilde{g}_i, \tilde{g}_j] = 0$, it follows that an involutive distribution always possesses a set of generators that are mutually commuting (those are given by $\bar{g}_j = \sum_{i=1}^m g_i \beta^i_j$).

For a vector field $f \in V^\infty(\mathcal{X})$ and a distribution $\mathcal{D}$ we define a new distribution $[f, \mathcal{D}] = \operatorname{span}\{[f, g], \forall\, g \in \mathcal{D}\}$ defined by all Lie brackets between f and $g \in \mathcal{D}$. If $\mathcal{D}$ is spanned by $m \ge 1$ independent vector fields g_i and $f \ne 0$, then we have $[f, \mathcal{D}] = \operatorname{span}\{[f, g_i], 1 \le i \le m\}$. For two distributions $\mathcal{D}_1$ and $\mathcal{D}_2$, we define their Lie bracket by $[\mathcal{D}_1, \mathcal{D}_2] = \operatorname{span}\{[f, g], f \in \mathcal{D}_1, g \in \mathcal{D}_2\}$ and their sum by $(\mathcal{D}_1 + \mathcal{D}_2)(x) = \mathcal{D}_1(x) + \mathcal{D}_2(x)$ (the sum in the right hand side is the usual sum of vector spaces). Given a distribution $\mathcal{D}$ we construct its associated Lie flag as

$$\mathcal{D}_0 := \mathcal{D}, \text{ and, iteratively, } \mathcal{D}_{i+1} = \mathcal{D}_i + [\mathcal{D}_0, \mathcal{D}_i], \quad \text{for any } i \ge 0.$$

Obviously we have $\mathcal{D}_i \subset \mathcal{D}_{i+1}$ for all $i \ge 0$. If all distributions $\mathcal{D}_i$, for $i \ge 0$, are of constant rank, then we define the *grow vector* of $\mathcal{D}$ as

$$\text{g.v.}\,(\mathcal{D}) = (d_0, d_1, \dots, d_{i^*}, d_{i^*+1}),$$

where $d_i = \operatorname{rk} \mathcal{D}_i$ and i^* is the smallest integer such that $d_{i^*} = d_{i^*+1}$. In particular, notice that $\mathcal{D}_{i^*}$ is involutive and of constant rank. We call $\mathcal{D}_{i^*}$ the *involutive closure* of $\mathcal{D}$ (it is unique) and denote it by $\bar{\mathcal{D}}$.

Given a distribution $\mathcal{D}$ we construct an involutive subdistribution (but, in general, it is not of constant rank) called the *characteristic distribution* and defined by

$$\mathcal{C}(\mathcal{D}) = \{g \in \mathcal{D}, \quad [g, \mathcal{D}] \subset \mathcal{D}\}.$$

Systems of first order PDEs. Consider a system of m first order linear partial differential equations:

$$\mathrm{L}_{g_i}(\lambda) = b_i, \qquad 1 \le i \le m, \tag{1.1}$$

for an unknown function $\lambda : \mathcal{X} \to \mathbb{R}$, where $g_i \in V^\infty(\mathcal{X})$, and $b_i \in C^\infty(\mathcal{X})$. Suppose that the distribution $\mathcal{D} = \operatorname{span}\{g_1, \dots, g_m\}$ is involutive and of constant rank m (i.e. the fields g_i's are independent). Then, define structure functions ν^k_{ij} by

$$[g_i, g_j] = \sum_{k=1}^m \nu^k_{ij}\, g_k, \quad \forall\, 1 \le i, j \le m.$$

The celebrated Frobenius theorem gives necessary and sufficient conditions for the (local) existence of a smooth solution λ of the system (1.1).

Theorem 1.2 (Frobenius)**.** *Let $\mathcal{D} = \operatorname{span}\{g_1, \dots, g_m\}$ be an involutive distribution of rank m. System (1.1) possesses a local solution λ if and only if the following integrability conditions are satisfied:*

$$\mathrm{L}_{g_i}(b_j) - \mathrm{L}_{g_j}(b_i) = \sum_{k=1}^{m} \nu_{ij}^k\, b_k, \qquad \forall\, 1 \le i < j \le m. \tag{1.2}$$

If $b_1 = \ldots = b_m = 0$, then the integrability conditions (1.2) are always satisfied, thus the homogeneous system $\mathrm{L}_{g_i}(\lambda) = 0$, for $1 \le i \le m$, admits solutions $\lambda(x)$ which have a direct interpretation. Namely, $S_c = \{\lambda(x) = c,\ c \in \mathbb{R}\}$ are m-dimensional integral leaves of the involutive distribution $\mathcal{D}$, that is, $T_x S_c = \mathcal{D}(x)$, hence proving that involutivity is equivalent to integrability.

Brief historical note. We shortly outline the story of the development of the theory of linear partial differential equations. The story begin by the study of linear partial differential equations (i.e. systems of the form (1.1) with $m = 1$) in two variables only (motivated by celestial mechanics, we believe). The idea of integrating first order partial differential equations is due to Euler in 1734 [Eul40]. Subsequent development were then made by Lagrange, Charpit, and Monge creating what we now call the method of characteristics [Lag67]. In 1815, Pfaff generalises Charpit's method to linear partial differential equations with any number of variables, his method was simplified by Cauchy in 1819 [Cau82] and by Jacobi in 1834 [Jac81] a century after Euler. Jacobi seems to be the first interested in systems of linear partial differential equations. The above theorem, attributed to Frobenius, appears to have been first proven (necessary conditions) in the homogeneous case (i.e. for $b_i = 0$) by Clebsch (see [Cle66] original work, and [Haw05] for an English version) and by Deahna for sufficient conditions [Dea40]. Frobenius is responsible for applying this theorem to differential forms and Pfaffian equations [Fro77], and thus definitely attached his name with it. The inhomogeneous case is not well developed in the literature because its proof roughly amounts to a reduction to a homogeneous case; in [Sal99] a proof is given in the case when the fields are mutually commuting (i.e. $[g_i, g_j] = 0$ for all $1 \le i, j \le m$). To our surprise, we have not found any modern text containing a proof of this theorem. See [Sam01], [Sal31, chapter 1], and references therein for a detailed historical introduction.

Pseudo-Riemannian geometry. A *pseudo-Riemannian* metric on a smooth n-dimensional manifold $\mathcal{X}$ is a map that smoothly associates to each point $x \in \mathcal{X}$ a non-degenerate symmetric bilinear form $\mathsf{g}_x(\cdot,\cdot)$ on the tangent space $T_x\mathcal{X}$ (see [Oli02] for a detailed introduction). Like in the classical Riemannian geometry, it is possible to define the Levi-Civita connection ∇ and the curvature $(0,4)$-tensor

$$\operatorname{Riem}(\mathsf{g})(X_1, X_2, X_3, X_4) = \mathsf{g}\left(R(X_1, X_2)X_3, X_4\right),$$

where $R(X_1, X_2)X_3 = \nabla_{X_1}\nabla_{X_2}X_3 - \nabla_{X_2}\nabla_{X_1}X_3 - \nabla_{[X_1,X_2]}X_3$, for any pseudo-Riemannian metric (see e.g. [Bes08]). A metric is called *flat* if its curvature tensor $\operatorname{Riem}(\mathsf{g})$ vanishes identically. Any pseudo-Riemannian g induces two mutually inverse isomorphisms

$$\sharp : T^*\mathcal{X} \to T\mathcal{X}, \quad \text{and} \quad \flat : T\mathcal{X} \to T^*\mathcal{X},$$

defined by $\mathsf{g}(\omega^\sharp, X) = \omega(X)$ and $X^\flat(Y) = \mathsf{g}(X, Y)$. Given two symmetric $(0,2)$-tensors g and h, we define their Kulkarni-Nomizu product $\mathsf{g} \owedge \mathsf{h}$, which is a $(0,4)$-tensor, by

$$\begin{aligned}\mathsf{g} \owedge \mathsf{h}(X_1, X_2, X_3, X_4) = \mathsf{g}(X_1, X_3)\mathsf{h}(X_2, X_4) + \mathsf{g}(X_2, X_4)\mathsf{h}(X_1, X_3) \\ - \mathsf{g}(X_1, X_4)\mathsf{h}(X_2, X_3) - \mathsf{g}(X_2, X_3)\mathsf{h}(X_1, X_4),\end{aligned}$$

for any X_1, X_2, X_3, and X_4 in $T\mathcal{X}$. We call two metrics g and g' to be *conformally equivalent* if there exists a smooth function $\lambda \neq 0$ such that $\mathsf{g}' = \lambda^2 \mathsf{g}$. A metric g is called conformally flat if it is conformally equivalent to a flat metric (in the above sense). In the case $\dim \mathcal{X} = 2$, all metrics are conformally flat, for $n \geq 3$ necessary and sufficient conditions for conformal flatness are given by the vanishing of the Cotton tensor (if $n = 3$) and by the vanishing of the Weyl tensor if $n \geq 4$. Those tensors are defined as follows. First, we define the Ricci tensor, the scalar curvature, and the Schouten tensor, respectively, by

$$\begin{aligned}\mathrm{Ric}\,(\mathsf{g})\,(X_2, X_3) = \mathrm{Tr}\,(X_1 \mapsto R(X_1, X_2)X_3)\,, \qquad s(\mathsf{g}) = \mathrm{Tr}_{\mathsf{g}}\mathrm{Ric}\,(\mathsf{g})\,, \\ \text{and} \quad P(\mathsf{g}) = \frac{1}{n-2}\left(\mathrm{Ric}\,(\mathsf{g}) - \frac{s(\mathsf{g})}{2(n-1)}\mathsf{g}\right),\end{aligned}$$

where Tr_{g} is the trace operator with respect to the metric, i.e. in local coordinates, with the components of the Ricci tensor denoted Ric_{ij}, we have $s(\mathsf{g}) = \sum_{i,j} \mathsf{g}^{ij}\mathrm{Ric}_{ij}$. Second, the Cotton and Weyl tensors are given, respectively, by

$$\begin{aligned}C(\mathsf{g})_{ijk} &= \nabla_k(\mathrm{Ric}_{ij}) - \nabla_j(\mathrm{Ric}_{ik}) + \frac{1}{4}\left(\nabla_j(s)\mathsf{g}_{ik} - \nabla_k(s)\mathsf{g}_{ij}\right), \\ \mathrm{Weyl}(\mathsf{g}) &= \mathrm{Riem}\,(\mathsf{g}) - P(\mathsf{g}) \owedge \mathsf{g}.\end{aligned}$$

For any smooth function f, we define its gradient $\mathrm{grad}\,(f) \in T\mathcal{X}$ by $(\mathrm{d}f)^\sharp$, i.e. for all $X \in T\mathcal{X}$, we have $\mathsf{g}(\mathrm{grad}\,(f)\,, X) = \mathrm{L}_X\,(f)$, and its hessian $\mathrm{Hess}\,(f) \in T^*\mathcal{X} \otimes T^*\mathcal{X}$ by

$$\mathrm{Hess}\,(f)\,(X_1, X_2) = \mathsf{g}(\nabla_{X_1}\mathrm{grad}\,(f)\,, X_2) = \mathrm{L}_{X_1}\,(\mathrm{L}_{X_2}\,(f)) - \mathrm{d}f(\nabla_{X_1}X_2).$$

If g and g' are two pseudo-Riemannian metrics related by $\mathsf{g}' = e^{2\phi}\mathsf{g}$, then their curvature tensors $\mathrm{Riem}\,(\mathsf{g}')$ and $\mathrm{Riem}\,(\mathsf{g})$ are related by

$$\mathrm{Riem}\,(\mathsf{g}') = e^{2\phi}\left(\mathrm{Riem}\,(\mathsf{g}) - \mathsf{g} \owedge \left(\mathrm{Hess}\,(\phi) - \mathrm{d}\phi \otimes \mathrm{d}\phi + \frac{1}{2}\|\mathrm{grad}\,(\phi)\,\|^2\mathsf{g}\right)\right), \tag{1.3}$$

where $\|X\|^2 = \mathsf{g}(X, X)$.

1.2 Bilinear algebra

One of our major construction will be a smooth version of some well-known classical bilinear algebra results, thus we review them here (see [Per98; Bou89] for a deeper introduction). Let E be an $\mathbb{R}$-vector space. We call a *bilinear* form on E a map $f : E \times E \to \mathbb{R}$ satisfying: both $f_x : y \mapsto f(x, y)$ and $f_y : x \mapsto f(x, y)$ are linear with respect to their argument. If E is n-dimensional equipped with a basis $e_1, \ldots, e_n$, then the form f is determined by n^2 numbers $a_{ij} = f(e_i, e_j)$. Indeed, if $A \in \mathrm{Mat}(n, \mathbb{R})$ denotes the matrix of the a_{ij}'s and if we set $x = \sum_{i=1}^n x_i e_i$ and $y = \sum_{j=1}^n y_j e_j$, then we have

$$f(x, y) = \sum_{i,j=1}^{n} a_{ij}\, x_i y_j.$$

If we adopt the convention to interpret x and y as column vectors whose components are x_i and y_j, respectively, we have the matrix notation

$$f(x, y) = x^t A y.$$

The matrix A is called the matrix of f relatively to the basis $e_1, \ldots, e_n$. We say that f is nondegenerate if $\det A \neq 0$. Observe that if we change the basis with $P \in GL_n(\mathbb{R})$ then the new matrix of f is $A' = P^t A P$, thus $\det A' = \delta^2 \det A$ with $\delta = \det P$, $\delta \neq 0$. We call f a *symmetric* bilinear form if $\forall\, x, y \in E \ f(x, y) = f(y, x)$. To any symmetric bilinear form we attach a map $F : E \to \mathbb{R}$ defined by $F(x) = f(x, x)$ and call it the *quadratic form* associated to f. Conversely, f is the *polar* form of F. We can pass from F to f by the following formula

$$f(x, y) = \frac{1}{2}\left[F(x + y) - F(x) - F(y)\right]. \tag{1.4}$$

Hence, in what follows, we shall replace f by F (or the other way around) and thus confuse their properties. A non-zero element $x \in E$ is called *isotropic* if $F(x) = 0$. We say that a linear subspace $V \subset E$ is *totally isotropic* if $F_{|V} = 0$. We call the *index* of F (and thus of f) the integer ν giving the maximal dimension of its totally isotropic subspaces. If $\nu = 0$, then F and f are called *anisotropic*.

Orthogonal group. Assume that f is a nondegenerate symmetric bilinear form, we call an *isometry* of E with respect to f any automorphism $u \in GL(E)$ satisfying

$$\forall\, x, y \in E, \quad f(u(x), u(y)) = f(x, y).$$

Isometries define a subgroup of $GL(E)$ called orthogonal group and denoted by $O(f)$. If f is represented by matrix A and u by matrix U, then u is an isometry if and only if $U^t A U = A$. Obviously, we have $\det u^2 = 1$, therefore $\det u = \pm 1$. This leads to the following definition: the subgroup of $O(f)$ formed by isometries with determinant equal to 1 is normal and called the *special orthogonal* group. It is denoted $SO(f)$. We have the following characterisation of isometries.

Proposition 1.1. *An element $u \in GL(E)$ is an isometry if and only if it preserves the quadratic form F attached to f, i.e.*

$$\forall\, x \in E, \quad F(u(x)) = F(x).$$

The proof is straightforward due to relation (1.4).

Similitudes. Consider a nondegenerate symmetric bilinear form f and let $u \in GL(E)$ and $\lambda \in \mathbb{R}^*$. We say that u is a *similitude* (relatively to f) with ratio of similarity (or multiplier) λ if we have

$$\forall\, x, y \in E, \quad f(u(x), u(y)) = \lambda\, f(x, y).$$

Observe that if the ratio is $\lambda = 1$, then u is actually an isometry. The group of similitudes is denoted by $GO(f)$. The ratio of similarity λ is well determined by u and the application that assigns to u its ratio is a group homomorphism with values in $\mathbb{R}^*$. Hence we have the short exact sequence

$$1 \longrightarrow O(f) \longrightarrow GO(f) \xrightarrow{\lambda} \mathbb{R}^*$$

but, in general, λ is not onto. Among similtudes there are isometries and homotheties. Indeed, a homothety with multiplicator λ is a similitude with ratio λ^2. Nevertheless, in general, the isometries and homotheties do not generate the whole group of similitudes. If A is the matrix of f and U the one of u, then the map u is a similitude with ratio λ if and only if we have $U^t AU = \lambda A$. We also have the following geometrical characterisation of similitudes.

Proposition 1.2. *Consider $u \in GL(E)$. Then u is a similitude if and only if u preserves orthogonality relations, i.e.*

$$u \in GO(f) \Longleftrightarrow (\forall\, x, y \in E, \quad f(x, y) = 0 \Rightarrow f(u(x), u(y)) = 0)\,.$$

If the index ν of f is greater or equal than 1, then we also have that u is a similitude if and only if u preserves isotropic elements of E, i.e.

$$u \in GO(f) \Longleftrightarrow (\forall\, x \in E, \quad F(x) = 0 \Rightarrow F(u(x)) = 0)\,.$$

Classification and signature. Two symmetric bilinear forms, f and f', on E are called *equivalent* if there exists $p \in GL(E)$ such that we have

$$\forall\, x, y \in E, \quad f'(x, y) = f(p(x), p(y)).$$

If the matrices of f and f' in a basis of E are A and A', respectively, then it is equivalent to say that there exists an invertible matrix P such that $A' = P^t AP$. Necessary conditions for the equivalence of f and f' are $\operatorname{rk} A = \operatorname{rk} A'$, $\nu(f) = \nu(f')$, and $\det A' = \det A$ up to the multiplication by a positive real.

Theorem 1.3 (Sylvester's law of inertia)**.** *There are $n + 1$ equivalence classes of nondegenerate symmetric bilinear forms f on E. In a suitable basis, f has for matrix*

$$A = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix},$$

with the plus sign present p-times and the minus sign $(n-p)$-times. In other words, if $x = (x_1, \ldots, x_n)^t$ in that basis, then we have

$$F(x) = \sum_{i=1}^{p} (x_i)^2 - \sum_{i=p+1}^{n} (x_i)^2.$$

See [Per98, chapter 5] or any text book on algebra for a proof. The pair $(p, n-p)$ is called the *signature* of F.

1.3 Lie algebra

When dealing with infinitesimal symmetries of control systems (see definition in the next subsection), we will use a certain number of facts about Lie algebras that we recall here; the reader will find much more details on this topic in [FH13; Kna13; Hal15].

Definition 1.4 (Lie algebra)**.** A *n-dimensional real Lie algebra* is a n-dimensional real vector space $\mathfrak{L}$, equipped with a map $[\cdot,\cdot]$ from $\mathfrak{L}\times\mathfrak{L}$ into $\mathfrak{L}$ satisfying the following properties:

a) $[\cdot,\cdot]$ is $\mathbb{R}$-bilinear,

b) $[\cdot,\cdot]$ is skew-symmetric, i.e. $\forall\, X, Y \in \mathfrak{L}, \quad [X,Y] = -\,[Y,X]$,

c) $[\cdot,\cdot]$ satisfies the Jacobi identity, i.e.

$$\forall\, X, Y, Z \in \mathfrak{L}, \quad [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

Consider a n-dimensional Lie algebra $\mathfrak{L} = \mathrm{vect}_{\mathbb{R}}\,\{v_1,\ldots,v_n\}$, we define its commutativity table (relative to this particular basis) by

$$[v_i, v_j] = \sum_{k=1}^{n} C_{ij}^k\, v_k, \quad C_{ij}^k \in \mathbb{R}.$$

The reals C_{ij}^k are called structure constants. A map $\varphi : \mathfrak{L}_1 \to \mathfrak{L}_2$ between two Lie algebras (of the same dimension) is a Lie algebra isomorphism if and only if it is a linear isomorphism (justifying that $\mathfrak{L}_1$ and $\mathfrak{L}_2$ must have the same dimension) satisfying

$$\forall\, X, Y \in \mathfrak{L}_1, \quad \varphi\left([X,Y]_{\mathfrak{L}_1}\right) = [\varphi(X), \varphi(Y)]_{\mathfrak{L}_2}\,.$$

In particular, a Lie algebra isomorphism need not preserve the structure constants. Nevertheless we have:

Proposition 1.3. *Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be two n-dimensional Lie algebras. If they both have a basis with respect to which the structure constants coincide, then $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are isomorphic.*

See [Bow05, Proposition 1] for a proof. A subset $\mathfrak{l}$ of a Lie algebra is a subalgebra if for all $l_1, l_2 \in \mathfrak{l}$ we have $[l_1, l_2] \in \mathfrak{l}$, if additionnaly $[l_1, l_2] = 0$ then $\mathfrak{l}$ is called an abelian subalgebra. A subset $\mathfrak{I}$ of a Lie algebra $\mathfrak{L}$ is an ideal if for any $w \in \mathfrak{I}$ and all $v \in \mathfrak{L}$ we have $[w, v] \in \mathfrak{I}$ (so, in particular, $\mathfrak{I}$ is a subalgebra). Given an ideal $\mathfrak{I}$ of a Lie algebra $\mathfrak{L}$ one constructs the quotient algebra $\mathfrak{L}/\mathfrak{I}$ as the usual quotient of vector spaces together with the Lie bracket defined by $[X+\mathfrak{I}, Y+\mathfrak{I}] = [X,Y]+\mathfrak{I}$.

1.4 Control Systems

In this subsection, we define control systems, that is, dynamical systems with additional parameters called controls. We will be interested in fully nonlinear and in control-affine systems. The former class is of the form

$$\Xi \,:\, \dot{x} = F(x, w), \quad x \in \mathcal{X}, \quad \text{and} \quad w \in \mathcal{W} \subset \mathbb{R}^m,$$

where x is the state of the system ($\mathcal{X}$ is a smooth n-dimensional manifold called the state space), $w = (w_1, \ldots, w_m)^t$ is the control taking values in $\mathcal{W}$, an open subset of $\mathbb{R}^m$, $F : \mathcal{X} \times \mathcal{W} \to \mathcal{X}$ is a smooth map, and the dot designs the derivative with respect to an independent variable, generally denoted t and representing time. A control-nonlinear system is, therefore, a system of nonlinear differential equations describing the temporal evolution of the state of a dynamical system under the action of a finite number of independent variables (controls) that can be freely chosen in order to achieve certain objectives. The second class of control-affine system is defined by

$$\Sigma : \dot{\xi} = f(\xi) + \sum_{i=1}^{m} g_i(\xi) u_i = f(\xi) + g(\xi) u,$$

where $g = (g_1, \ldots, g_m)$ and $u = (u_1, \ldots, u_m)^t$. The state ξ belongs to a smooth manifold $\mathcal{M}$, in our work $\mathcal{M}$ will be different from $\mathcal{X}$, and the control u takes values in $\mathbb{R}^m$. A control-affine system is given by a $(m+1)$-tuple of vector fields $(f, g_1, \ldots, g_m)$ and will be denoted by the pair $\Sigma = (f, g)$. To any control-affine system $\Sigma = (f, g)$ we will attach the following distributions

$$\begin{aligned} \mathcal{D}^0 &= \operatorname{span}\{g_1, \ldots, g_m\}, \\ \mathcal{D}^1 &= \mathcal{D}^0 + [f, \mathcal{D}^0] = \operatorname{span}\{g_1, \ldots, g_m, \operatorname{ad}_f g_1, \ldots, \ldots, \operatorname{ad}_f g_m\}. \end{aligned} \tag{1.5}$$

Feedback equivalence. Two control-nonlinear systems Ξ and $\tilde{\Xi}$, resp. control-affine systems $\Sigma = (f, g)$ and $\tilde{\Sigma} = (\tilde{f}, \tilde{g})$, are called state-equivalent if they are related by a diffeomorphism of the state space, that is, $\exists\, \phi : \mathcal{X} \to \tilde{\mathcal{X}}$ such that

$$\begin{aligned} &\phi_* F(\cdot, w) = \tilde{F}(\cdot, w), \quad \forall\, w \in \mathcal{W} \quad \text{in the nonlinear case,} \\ &\phi_*(f) = \tilde{f} \quad \text{and} \quad \phi_*(g_i) = \tilde{g}_i, \quad \forall\, 1 \le i \le m \quad \text{in the control-affine case.} \end{aligned}$$

Consequently, their trajectories (corresponding to the same controls) will be related by that diffeomorphism. When considering state-equivalence the controls remain unchanged. Feedback equivalence enlarges the previous transformations by permitting transformation of controls as well.

Two control-nonlinear systems $\Xi : \dot{x} = F(x, w)$ and $\tilde{\Xi} : \dot{\tilde{x}} = \tilde{F}(\tilde{x}, \tilde{w})$ are called *feedback equivalent* if there exists a diffeomorphism $\Phi : \mathcal{X} \times \mathcal{W} \to \tilde{\mathcal{X}} \times \tilde{\mathcal{W}}$ of the form

$$(\tilde{x}, \tilde{w}) = \Phi(x, w) = (\phi(x), \psi(x, w)),$$

which transforms the first system into the second, i.e.

$$D\phi(x) F(x, w) = \tilde{F}(\phi(x), \psi(x, w)).$$

Notice that the diffeomorphism Φ is triangular: indeed, the map ϕ depends on x only and plays the role of a coordinates change in the state space $\mathcal{X}$, and the map ψ, called feedback, changes the parametrisation by control w in a way that depends on the state x. The trajectories of both systems coincide, however, they are differently parametrised with respect to controls. If the diffeomorphism Φ is defined in a neighbourhood of (x_0, w_0) only, and $\Phi(x_0, w_0) = (\tilde{x}_0, \tilde{w}_0)$, then we say that the two systems are *locally feedback equivalent* at (x_0, w_0) and $(\tilde{x}_0, \tilde{w}_0)$, respectively. If Φ is defined locally around x_0 and globally with respect to w, i.e. it maps $\mathcal{X}_0 \times \mathcal{W}$ into $\tilde{\mathcal{X}}_0 \times \tilde{\mathcal{W}}$, then we say that Ξ and $\tilde{\Xi}$ are locally feedback equivalent at x_0 and $\tilde{x}_0$, respectively.

For control-affine systems of the form $\Sigma : \dot{\xi} = f(\xi) + g(\xi)u$, in order to preserve the affine shape of the system, feedback transformations are restricted to those of the form

$$\tilde{u} = \psi(\xi, u) = \alpha(\xi) + \beta(\xi)u,$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)^t$ and $\beta = (\beta^i_j)$ are smooth functions depending on the state and satisfy $\beta(\cdot) \in GL_m(\mathbb{R})$. Then the feedback equivalence of $\Sigma = (f, g)$ and $\tilde{\Sigma} = (\tilde{f}, \tilde{g})$ means that

$$\phi_* (f) = \tilde{f} + \tilde{g}\alpha \quad \text{and} \quad \phi_* (g) = \tilde{g}\beta.$$

In that case, we denote the feedback transformation by the triple (ϕ, α, β) and if $\phi = \mathrm{Id}$, then this action is called a *pure feedback* transformation and is denoted (α, β). To any control-affine system Σ we can attached the affine distribution

$$\mathcal{A} = f + \operatorname{span}\{g_1, \ldots, g_m\},$$

and feedback equivalence of two control-affine systems $\Sigma = (f, g)$ and $\tilde{\Sigma} = (\tilde{f}, \tilde{g})$ simply means equivalence of the corresponding affine distribution, $\phi_*\mathcal{A} = \tilde{\mathcal{A}}$. A point ξ_0 is an equilibrium of a control-affine system Σ if $0 \in \mathcal{A}(\xi_0)$. For any equilibrium point ξ_0 there exists a control $u_0 \in \mathbb{R}^m$ such that $f(\xi_0) + g(\xi_0)u_0 = 0$; moreover, u_0 is unique if $\dim \mathcal{D}^0(\xi_0) = m$. It is a straightforward calculation to see that if the distribution $\mathcal{D}^0$ is involutive, then both $\mathcal{D}^0$ and $\mathcal{D}^1$ (defined in (1.5)) are invariant under the action of feedback transformations (ϕ, α, β).

Infinitesimal symmetries. We say that a diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$ is a *symmetry* of a control-affine system $\Sigma = (f, g)$ if it preserves the field of affine m-planes $\mathcal{A}$ (equivalently, the affine distribution $\mathcal{A}$), that is,

$$\phi_*\mathcal{A} = \mathcal{A}.$$

We say that a vector field v on $\mathcal{M}$ is an infinitesimal symmetry of Σ if the (local) flow γ^v_t of v is a local symmetry, for any t for which it exists, that is, $(\gamma^v_t)_*\mathcal{A} = \mathcal{A}$. Consider the system $\Sigma = (f, g)$ and recall that $\mathcal{D}^0 = \operatorname{span}\{g_1, \ldots, g_m\}$ is the distribution spanned by the vector fields g_i. We have the following characterisation of infinitesimal symmetries (see [RT04; GM85] for a more detailed introduction), whose proof is straightforward.

Proposition 1.4. *A vector field v is an infinitesimal symmetry of the system $\Sigma = (f, g)$ if and only if $[v, g_i] = 0 \mod \mathcal{D}^0$, for $1 \le i \le m$, and $[v, f] = 0 \mod \mathcal{D}^0$.*

By the Jacobi identity, it is easy to see that if v_1 and v_2 are infinitesimal symmetries, then so is $[v_1, v_2]$, hence the set of all infinitesimal symmetries forms a Lie algebra (not necessarily of finite dimension). Notice that the Lie algebra of infinitesimal symmetries is attached to the affine distribution $\mathcal{A}$ and not to different $(m+1)$-tuples $(f, g_1, \ldots, g_m)$ defining $\mathcal{A}$ that are related to each other via feedback transformations (ϕ, α, β).

Prolongations. Any control-nonlinear system $\Xi : \dot{x} = F(x, w)$ can be prolonged to a control-affine system Ξ^p by augmenting the state space with the controls $w = (w_1, \dots, w_m)^t$ and introducing new controls $u_i = \dot{w}_i$, for $i = 1, \dots, m$, which gives,

$$\Xi^p : \begin{cases} \dot{x} &= F(x, w) \\ \dot{w} &= u \end{cases}, \quad u \in \mathbb{R}^m.$$

It appears that the idea of prolonging a control-nonlinear system was introduced by van der Schaft in [van82, Definition 3.4] (prolongations are called *extended systems* in his work) and then used to study controllability and observability of nonlinear systems in [van84]. Observe that Ξ^p has as its state space $\mathcal{M} = \mathcal{X} \times \mathcal{W}$, a smooth manifold of dimension $n^p = n + m$, and is given by vector fields $f = F(x, w)\frac{\partial}{\partial x}$ and $g_i = \frac{\partial}{\partial w_i}$ for $i = 1, \dots, m$. In particular, for that system, the distribution $\mathcal{D}^0 = \operatorname{span}\{g_1, \dots, g_m\}$ is involutive and of constant rank m. Conversely, any control-affine system of the form Ξ^p can be reduced to a control-nonlinear system simply by removing the w-components and then treating w as a control. The following result characterises control-affine systems Σ that are feedback equivalent to the prolongation Ξ^p of a control-nonlinear systems Ξ; in other words, it shows when a control-affine system Σ can be reduced to a nonlinear system Ξ whose state space is smaller (by paying the price of nonlinearity).

Proposition 1.5. *A control-affine system $\Sigma = (f, g)$, defined on a n^p-dimensional manifold with $1 \le m < n^p$ controls, is feedback equivalent to the prolongation Ξ^p of a control-nonlinear system Ξ, defined on a $(n^p - m)$-dimensional manifold with m controls, if and only if the distribution $\mathcal{D}^0$ of Σ is involutive and of constant rank m.*

Flavour of the proof: Necessity is obvious by the above considerations and the fact that involutivity of $\mathcal{D}^0$ is preserved by feedback transformations. Conversely, using Frobenius theorem (see Theorem 1.1), we restrict the action of the control along the last m components of the coordinates and then we remove them.

2 Description of the problems

In this section, we introduce the problem that we consider in this first part of the thesis. We first begin by stating the problem in its generality, then we explain how we will deal with it by introducing an equivalent formulation.

Let $\mathcal{X}$ be a smooth connected manifold of dimension $n \ge 2$, equipped with local coordinates x. In the tangent bundle $T\mathcal{X}$, equipped coordinates $(x, \dot{x})$, we consider a smooth $(2n-1)$-dimensional submanifold $\mathcal{S}$ (a hypersurface) given by

$$\mathcal{S} = \{(x, \dot{x}) \in T\mathcal{X},\ S(x, \dot{x}) = 0\},$$

where $S : T\mathcal{X} \to \mathbb{R}$ is a smooth scalar function satisfying $\operatorname{rk} \frac{\partial S}{\partial \dot{x}} = 1$ for all $(x, \dot{x}) \in \mathcal{S}$. Observe that the submanifold $\mathcal{S}$ defines an implicit ordinary differential equation given by $S(x, \dot{x}) = 0$, see [Olv95, Chapter 6] for a similar construction in the space of k-jets. From this point of view, a smooth solution of the given differential equation is a smooth curve $I \ni t \mapsto x(t) \in \mathcal{X}$ such that $S(x(t), \dot{x}(t)) = 0$ holds for any $t \in I$.

This is equivalent to the statement that the graph of $(x(t), \dot{x}(t)) \in T\mathcal{X}$ must lie entirely within the submanifold determined by the differential equation, i.e.

$$\{(x(t), \dot{x}(t)),\ t \in I\} \subset \mathcal{S}.$$

We now explain what it means for two submanifold of $T\mathcal{X}$ to be equivalent.

Definition 1.5 (Equivalence of submanifolds)**.** Two submanifolds $\mathcal{S} \subset T\mathcal{X}$ and $\tilde{\mathcal{S}} \subset T\tilde{\mathcal{X}}$ are equivalent if there exists a diffeomorphism $\phi : \mathcal{X} \to \tilde{\mathcal{X}}$ and a smooth nonvanishing function $\delta : T\mathcal{X} \to \mathbb{R}$ such that

$$\tilde{S}(\phi(x), D\phi(x)\dot{x}) = \delta(x, \dot{x})\, S(x, \dot{x}), \tag{1.6}$$

where $D\phi$ is the derivative (tangent map) of ϕ.

This definition implies that for all $(x, \dot{x}) \in \mathcal{S}$ we have $(\tilde{x}, \dot{\tilde{x}}) = (\phi(x), D\phi(x)\dot{x}) \in \tilde{\mathcal{S}}$, hence the diffeomorphism point transformation $(\phi, D\phi)$ maps the graph of $S^{-1}(0)$ into that of $\tilde{S}^{-1}(0)$. Equivalence of submanifolds $\mathcal{S}$ and $\tilde{\mathcal{S}}$ means simply that implicit underdetermined ordinary differential equations $S(x, \dot{x}) = 0$ and $\tilde{S}(\tilde{x}, \dot{\tilde{x}}) = 0$ are equivalent; a similar definition can be found in [Bog14, Definition 2].

Example. Since rk $\frac{\partial S}{\partial \dot{x}} = 1$, we locally have a splitting of the coordinates $x = (z, y)$, with $\dim(z) = 1$, such that $\frac{\partial S}{\partial \dot{z}}(x_0, \dot{x}_0) \neq 0$. By Malgrange preparation theorem (actually, the implicit function theorem) [Mal62], we can write $S(x, \dot{x}) = \delta(x, \dot{x})\,(\dot{z} - s(x, \dot{y}))$, where $\delta(x_0, \dot{x}_0) \neq 0$. Hence, the submanifold $\mathcal{S}$ is locally equivalent to the one given by $\dot{z} - s(x, \dot{y}) = 0$. •

It is natural to ask how to characterise and classify submanifolds $\mathcal{S}$ of certain particular classes, for instance the class of linear submanifolds $\mathcal{S}_{lin}$ given by $S_{lin}(x, \dot{x}) = \omega(x)\dot{x} = 0$ or the more general class of affine submanifolds $\mathcal{S}_{aff}$ given by $S_{aff}(x, \dot{x}) = \omega(x)\dot{x} + h(x) = 0$, where ω is a differential one-form on $\mathcal{X}$ (satisfying $\omega(x_0) \neq 0$) and h is a smooth function on $\mathcal{X}$. The name *linear*, resp. *affine*, comes from the fact that S_{lin}, resp. S_{aff}, is linear, resp. affine, with respect to the velocities $\dot{x}$, in other words, in each fiber $T_x\mathcal{X}$, the submanifold $\mathcal{S}_{lin}$, resp. $\mathcal{S}_{aff}$ is a linear, resp. an affine, subspace. Those questions have been widely studied under the prism of Pfaffian equations (linear and affine) and go back to Pfaff, Darboux, and Cartan [Pfa15; Dar82; Car31]. Although the problem of classification of Pfaffian equations is still open in its full generality, many important results have been obtained for various classes of linear Pfaffian equations (contact and quasi-contact case, Martinet case, singularities, [Zhi92; Zhi95; Mar70; JZ01; Elk12]) and of affine Pfaffian equations(dimension two [JR90], three [RZ95; Res98], and arbitrary [ZR98; Elk12]). The problems addressed in this first part of the manuscript are the natural extension, of the above introductory questions of characterising affine $\mathcal{S}_{aff}$ and linear $\mathcal{S}_{lin}$ submanifolds, to a larger class of submanifolds of the tangent bundle. Namely, our purpose is to give a characterisation and a classification of submanifolds $\mathcal{S} \subset T\mathcal{X}$ such that in each fiber $T_x\mathcal{X}$, the hypersurface $\mathcal{S}_x$ describes a quadric, i.e. in a well chosen coordinate system, $\mathcal{S}_x$ is given by the zero-level set of a quadratic map with respect to the velocities.

Problem 1. Consider a smooth n-dimensional manifold $\mathcal{X}$. Give a characterisation of the equivalence (in the sense of Definition 1.5) of a submanifold $\mathcal{S} \subset T\mathcal{X}$ with a quadric $\mathcal{S}_q$ given by the zero-level set of the map

$$S_q(x, \dot{x}) = \dot{x}^t \mathsf{g}(x)\dot{x} + 2\omega(x)\dot{x} + h(x). \tag{1.7}$$

The map S_q can be represented by the triple (g, ω, h), where g is a smooth symmetric $(0,2)$-tensor (possibly degenerated), ω is a smooth differential one-form, and h is a smooth function. Clearly, two quadrics $\mathcal{S}_q$ of $T\mathcal{X}$ and $\tilde{\mathcal{S}}_q$ of $T\tilde{\mathcal{X}}$, given by (g, ω, h) and $(\tilde{\mathsf{g}}, \tilde{\omega}, \tilde{h})$, respectively, are equivalent if and only if there exists a diffeomorphism $\tilde{x} = \phi(x)$ and a non-vanishing function $\delta = \delta(x)$ on $\mathcal{X}$ such that $\delta \mathsf{g} = \phi^* \tilde{\mathsf{g}}$, $\delta\omega = \phi^*\tilde{\omega}$, and $\delta h = \phi^* \tilde{h}$. In particular, observe that the tensors g and $\tilde{\mathsf{g}}$, which are possibly degenerated (pseudo-)Riemanian metrics, are conformally equivalent. Compare with [Tar11, definition 2].

We now give a brief description of the class of quadric submanifolds. It is well-known in affine geometry that quadratic equations can be classified by the rank and signature (up to order) of the matrix $M_q(x) = \begin{pmatrix} \mathsf{g}(x) & \omega(x)^t \\ \omega(x) & h(x) \end{pmatrix}$ and that of $\mathsf{g}(x)$; see e.g. [Tar11, Corollary 8.14]. We will use the following two determinants

$$\Delta_1(x) = \det(M_q(x)) \quad \text{and} \quad \Delta_2(x) = \det(\mathsf{g}(x)).$$

Of course, Δ_1 and Δ_2 depend on the choice of coordinates but the ideals generated by them (and thus their zero-level set) are invariantly related to the submanifold $\mathcal{S}_q$. Therefore, the rank and the signature of $M_q(x)$ and that of $\mathsf{g}(x)$ do not depend on the coordinates and, moreover, are constant in a neighbouhood of x_0 if $\Delta_1(x_0) \neq 0$ and $\Delta_2(x_0) \neq 0$. Under the multiplication by a non-vanishing function $\delta(x)$, the rank of $M_q(x)$ of that of $\mathsf{g}(x)$ are clearly preserved, and their signature (up to order) is also preserved. In this work we will characterise non-degenerate quadrics, that is, non-empty and satisfying $\Delta_1 \neq 0$. The non-degeneracy assumption $\Delta_1 \neq 0$ implies that $\operatorname{rk} \mathsf{g}(x) \geq n-1$.

Remark (Conic sections). The simplest example of quadrics are those given in a space of dimension $n = 2$, they are called conics. There are three non-degenerated conics: the ellipse, the hyperbola (both corresponding to $\Delta_2 \neq 0$), and the parabola corresponding to $\Delta_2 = 0$. Those are studied in Chapter 2 of the manuscript. ◆

Remark (Non-degenerate real quadric surfaces). In dimension $n = 3$, there are only five non-degenerated quadrics; namely ellipsoid, hyperboloid of one or two sheets (corresponding to $\Delta_2 \neq 0$), and elliptic or hyperbolic paraboloid (corresponding to $\Delta_2 = 0$). The latter two cases are studied in Chapter 4 of the manuscript. ◆

While the first part of the manuscript is entirely dedicated to the study of quadric submanifold, only in the lowest dimension ($n = 2$) we are able to give a complete theory, that is to treat all conics as one general result. In higher dimensions, we will focus on the following case: we assume that $\operatorname{rk} \mathsf{g} = n - 1$ in a neighbourhood (i.e. the degenerate case for a non-degenerate quadric). That assumption corresponds for $n = 2$ to the class of parabolas, and for $n = 3$ to the class of elliptic and hyperbolic paraboloids. Observe that the non-degeneracy condition $\Delta_1 \neq 0$ implies that $\langle \omega, A \rangle \neq 0$ for any non-zero vector field $A \in \ker \mathsf{g}$. Therefore, in a suitable coordinate system, S_q takes the form

$$S_Q(x, \dot{x}) = \dot{z} - \dot{y}^t Q(x) \dot{y} - b(x)\dot{y} - c(x), \tag{1.8}$$

where $x = (z, y)$, with $y = (y_1, \ldots, y_{n-1})$, such that $\ker \mathsf{g} = \operatorname{span}\left\{\frac{\partial}{\partial z}\right\}$, $Q(x)$ is a symmetric non-degenerate $(0,2)$-tensor with constant signature (p, q), $b(x)$ is a smooth one-form, and $c(x)$ is a smooth function. The submanifolds given by an equation of the form $S_Q(x, \dot{x}) = 0$ are called (p,q)-paraboloids, or shortly paraboloids, by analogy with the parabolic case when $n = 2$ and the terminology when $n = 3$; they will be denoted by the 4-tuple $\left(\frac{\partial}{\partial z}, Q, b, c\right)$.

After giving a characterisation of quadric submanifolds (all conics when $n = 2$ and paraboloids when $m \geq 3$) we will be interested in the classification problem, expressed as:

Problem 2. Describe and characterise some orbits of the quadric submanifolds given by (1.7) under the notion of equivalence described by Definition 1.5.

Taxonomy of paraboloid submanifolds. When $n \geq 3$, we will exclusively be interested in classifying paraboloid submanifolds $\mathcal{S}_Q = \left(\frac{\partial}{\partial z}, Q, b, c\right)$, defined by (1.8). It seems natural to seek conditions to normalise the symmetric $(0,2)$-tensor Q first, then conditions to normalise the differential one-form b, and finally conditions to normalise the function c.

The two chosen normalisations for Q are the following. First, we will characterise the submanifolds where Q is diagonal, i.e. we give conditions for the smooth diagonalisation, thus our result can be interpreted as a smooth version of the spectral theorem. Second, we will characterise the equivalence of Q to the constant symmetric matrix $\mathtt{I}_{p,q} = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}$, where (p,q) is the signature of Q, that result is similar to Sylvester's law of inertia. Of course, it is well-known from bilinear algebra that those normalisations can be done everywhere pointwise, however we will show that doing it smoothly require additional conditions (similarly to the conditions for the flatness of a metric tensor). Next, we present a normal form with b being normalised to 0. And finally we will normalise the function c to a real constant; the form with $c = 0$ will be special and called a null-form paraboloid submanifold. The following table summarise the chosen nomenclature used to denote different families of paraboloid submanifolds.

Name	Form
paraboloid submanifold	$\mathcal{S}_Q = \{\dot{z} = \dot{y}^t Q(x)\dot{y} + b(x)\dot{y} + c(x)\}$
diagonal paraboloid submanifold	$\mathcal{S}_Q^d = \{\dot{z} = \dot{y}^t D(x)\dot{y} + b(x)\dot{y} + c(x)\}$
weakly flat paraboloid submanifold	$\mathcal{S}_Q' = \{\dot{z} = \dot{y}^t \mathtt{I}_{p,q}\dot{y} + b(x)\dot{y} + c(x)\}$
strongly flat paraboloid submanifold	$\mathcal{S}_Q'' = \{\dot{z} = \dot{y}^t \mathtt{I}_{p,q}\dot{y} + c(x)\}$
constant-form paraboloid submanifold	$\mathcal{S}_Q''' = \{\dot{z} = \dot{y}^t \mathtt{I}_{p,q}\dot{y} + c\}$
null-form paraboloid submanifold	$\mathcal{S}_Q^0 = \{\dot{z} = \dot{y}^t \mathtt{I}_{p,q}\dot{y}\}$

Table 1: Nomenclature of paraboloid submanifolds.

The denomination of weakly and strongly flat paraboloid submanifolds will become clear when the conditions of their characterisation will be given.

Equivalent reformulation. Our analysis of problem 1 and problem 2 will be based on attaching to a submanifold $\mathcal{S} = \{S(x, \dot{x}) = 0\} \subset T\mathcal{X}$ two control systems.

First, a nonlinear one given by

$$\Xi_{\mathcal{S}} \,:\, \dot{x} = F(x,w), \quad x \in \mathcal{X}, \quad w \in \mathcal{W} \subset \mathbb{R}^{n-1},$$

where $\dot{x} - F(x,w) = 0$ is a regular parametric representation of $\mathcal{S}$, that is, for all $w = (w_1, \ldots, w_{n-1}) \in \mathcal{W}$ (interpreted as controls) we have $S(x, F(x,w)) = 0$ and $\mathrm{rk}\, \frac{\partial F}{\partial w}(x,w) = n-1$. Second, a control-affine one $\Sigma_{\mathcal{S}}$ given by the prolongation $\Xi^p_{\mathcal{S}}$ of $\Xi_{\mathcal{S}}$:

$$\Sigma_{\mathcal{S}} \,:\, \begin{cases} \dot{x} &= F(x,w) \\ \dot{w} &= u \end{cases}, \quad (x,w) \in \mathcal{M} = \mathcal{X} \times \mathcal{W}, \quad u \in \mathbb{R}^{n-1}.$$

They are called, respectively, a *first* and *second prolongation* of $\mathcal{S}$. Notice that, since we are interested in corank one submanifold of $T\mathcal{X}$, the number m of controls and the dimension n of the state space are related as follows. When considering a first prolongation we have $m = n-1$, and when considering a second prolongation we have $m = \frac{1}{2}(n-1)$. Observe that, since $\mathcal{S}$ relates the positions x with the velocities $\dot{x}$, it describes a nonholomic constraint. We say that a smooth curve $x(t) \in \mathcal{X}$ satisfies the nonholonomic constraint given by $\mathcal{S}$ if we have $(x(t), \dot{x}(t)) \in \mathcal{S}$. Clearly, $x(t)$ satisfies the nonholonomic constraint described by $\mathcal{S}$ (equivalently, satisfies the differential equation $S(x(t), \dot{x}(t)) = 0$) if and only if $x(t)$ is a trajectory of $\Xi_{\mathcal{S}}$ for a certain smooth control $w(t)$ or, equivalently, $(x(t), w(t))$ is a trajectory of $\Sigma_{\mathcal{S}}$ for a smooth control $u(t)$.

An observation that links studying submanifolds $\mathcal{S} \subset T\mathcal{X}$ and their prolongations $\Xi_{\mathcal{S}}$ and $\Sigma_{\mathcal{S}}$ is that equivalence of submanifolds corresponds to the equivalence of control systems $\Xi_{\mathcal{S}}$ and $\Sigma_{\mathcal{S}}$ via feedback transformations, as assured by the following result.

Proposition 1.6 (Equivalence of equivalence relations)**.** *Consider submanifolds $\mathcal{S}$ of $T\mathcal{X}$ and $\tilde{\mathcal{S}}$ of $T\tilde{\mathcal{X}}$. The following statements are locally equivalent:*

(i) The submanifolds $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are equivalent via $\phi(x)$ and $\delta(x,\dot{x})$.

(ii) Their regular parametrisations (first prolongations) $\Xi_{\mathcal{S}}$ and $\tilde{\Xi}_{\tilde{\mathcal{S}}}$ are feedback equivalent via $\phi(x)$ and $\psi(x,w)$.

(iii) Their second prolongations $\Sigma_{\mathcal{S}}$ and $\tilde{\Sigma}_{\tilde{\mathcal{S}}}$ are feedback equivalent via $\varphi(x,w) = (\phi(x), \psi(x,w))$ and (α,β).

That is, the following diagram is commutative:

$$\begin{array}{ccccc}
\mathcal{S} & \xrightarrow{\text{parametrisation}} & \Xi_{\mathcal{S}} & \xrightarrow{\text{prolongation}} & \Sigma_{\mathcal{S}} \\
{\scriptstyle(\phi,\delta)}\updownarrow & & {\scriptstyle(\phi,\psi)}\updownarrow & & \updownarrow{\scriptstyle(\varphi,\alpha,\beta)} \\
\tilde{\mathcal{S}} & \xrightarrow{\text{parametrisation}} & \tilde{\Xi}_{\tilde{\mathcal{S}}} & \xrightarrow{\text{prolongation}} & \tilde{\Sigma}_{\tilde{\mathcal{S}}}
\end{array}$$

Our proof will use the following lemma, which relates the regularity of $\mathcal{S}$ to the one of its first prolongation.

Lemma 1.1. *Let $\mathcal{X}$ be a smooth n-dimensional manifold equipped with coordinates $x = (z,y)$, with $y = (y_1, \ldots, y_{n-1})$. Consider a submanifold $\mathcal{S} = \{S(x,\dot{x}) = 0\}$ of $T\mathcal{X}$ together with any of its regular parametrisation $\Xi_{\mathcal{S}} : \begin{cases} \dot{z} &= F_1(x,w) \\ \dot{y} &= F_2(x,w) \end{cases}$. Then, $\frac{\partial S}{\partial \dot{z}} \neq 0$ if and only if $\mathrm{rk}\, \frac{\partial F_2}{\partial w} = n-1$.*

Proof. Consider the submanifold $\mathcal{S} = \{S(x,\dot{x}) = 0\}$ of the tangent bundle $T\mathcal{X}$ together with any of its regular parametrisation $\Xi_{\mathcal{S}} : \begin{cases} \dot{z} &= F_1(x,w) \\ \dot{y} &= F_2(x,w) \end{cases}$. In particular, for all w we have $S(x, F_1(x,w), F_2(x,w)) = 0$, which implies, by differentiating with respect to w, that it holds

$$\frac{\partial S}{\partial \dot{z}}\frac{\partial F_1}{\partial w} + \frac{\partial S}{\partial \dot{y}}\frac{\partial F_2}{\partial w} = 0. \tag{1.9}$$

Assume that $\frac{\partial S}{\partial \dot{z}} \neq 0$. If $\mathrm{rk}\,\frac{\partial F_2}{\partial w} < n-1$, then there exists v such that $\frac{\partial F_2}{\partial w}v = 0$, but by the last relation we conclude that $\frac{\partial F_1}{\partial w}v = 0$, a contradiction with the regularity of $\Xi_{\mathcal{S}}$ (i.e. with the assumption $\mathrm{rk}\begin{pmatrix} \frac{\partial F_1}{\partial w} \\ \frac{\partial F_2}{\partial w} \end{pmatrix} = n-1$).

Conversely, assume that $\mathrm{rk}\,\frac{\partial F_2}{\partial w} = n-1$. If $\frac{\partial S}{\partial \dot{z}} = 0$, then by equation (1.9) we conclude that $\frac{\partial S}{\partial \dot{y}} = 0$ a contradiction with the regularity of $\mathcal{S}$ (namely, $\frac{\partial S}{\partial \dot{x}} \neq 0$). ■

Proof of Proposition 1.6. (i)⇒(ii). Assume that two submanifolds $\mathcal{S}$ and $\tilde{\mathcal{S}}$ given, respectively, by the equations $S(x,\dot{x}) = 0$ and $\tilde{S}(\tilde{x},\dot{\tilde{x}}) = 0$ are equivalent via a diffeomorphism $\tilde{x} = \phi(x)$ and a nonvanishing function $\delta(x,\dot{x})$, that is $\tilde{S}(\phi(x), D\phi(x)\dot{x}) = \delta(x,\dot{x})S(x,\dot{x})$. Consider $\Xi_{\mathcal{S}} : \dot{x} = F(x,w)$ and $\tilde{\Xi}_{\tilde{\mathcal{S}}} : \dot{\tilde{x}} = \tilde{F}(\tilde{x},\tilde{w})$, two regular parametrisations of those submanifolds. Then, using $0 = S(x,\dot{x}) = S(x, F(x,w))$, we obtain

$$\tilde{S}(\phi(x), D\phi(x)F(x,w)) = \delta(x, F(x,w))S(x,F(x,w)) = 0, \quad \forall\, w \in \mathcal{W},$$

implying $\tilde{S}(\tilde{x}, \hat{F}(\tilde{x},w)) = 0$, where $\hat{F}(\tilde{x},w) = D\phi(\phi^{-1}(\tilde{x}))F(\phi^{-1}(\tilde{x}),w)$. Therefore, $\dot{\tilde{x}} = \tilde{F}(\tilde{x},\tilde{w})$ and $\dot{\tilde{x}} = \hat{F}(\tilde{x},w)$ are two regular parametrisations of the same submanifold $\tilde{\mathcal{S}}$. We will prove that $\tilde{F}(\tilde{x},\tilde{w})$ and $\hat{F}(\tilde{x},w)$ are related by an invertible (pure) feedback transformation of the form $\tilde{w} = \psi(x,w)$.

Without loss of generality assume that we have $\tilde{x} = (\tilde{z},\tilde{y})$ where $\frac{\partial \tilde{S}}{\partial \dot{\tilde{z}}} \neq 0$. Applying Lemma 1.1 to the submanifold $\tilde{\mathcal{S}} = \{\tilde{S}(\tilde{x},\dot{\tilde{x}}) = 0\}$ and to its two regular parametrisation

$$\tilde{\Xi}_{\tilde{\mathcal{S}}} : \begin{cases} \dot{\tilde{z}} &= \tilde{F}_1(\tilde{x},\tilde{w}) \\ \dot{\tilde{y}} &= \tilde{F}_2(\tilde{x},\tilde{w}) \end{cases} \quad \text{and} \quad \hat{\Xi}_{\tilde{\mathcal{S}}} : \begin{cases} \dot{\tilde{z}} &= \hat{F}_1(\tilde{x},w) \\ \dot{\tilde{y}} &= \hat{F}_2(\tilde{x},w) \end{cases}$$

yields $\mathrm{rk}\,\frac{\partial \tilde{F}_2}{\partial \tilde{w}}(\tilde{x}_0,\tilde{w}_0) = \mathrm{rk}\,\frac{\partial \hat{F}_2}{\partial w}(\tilde{x}_0,w_0) = n-1$. Therefore, we deduce that $\tilde{w} = \tilde{F}_2^{-1}(\tilde{x},\dot{\tilde{y}}) = \tilde{F}_2^{-1}(\tilde{x},\hat{F}_2(\tilde{x},w))$. By the implicit function theorem, we get that $\tilde{\mathcal{S}}$ is equivalent to $\dot{\tilde{z}} - \tilde{s}(\tilde{x},\dot{\tilde{y}}) = 0$, implying that $\tilde{F}_1(\tilde{x},\tilde{F}_2^{-1}(\tilde{x},\hat{F}_2(\tilde{x},w))) = \hat{F}_1(\tilde{x},w)$. Thus, $\tilde{\Xi}_{\tilde{\mathcal{S}}}$ and $\hat{\Xi}_{\tilde{\mathcal{S}}}$ are pure feedback equivalent via $\tilde{w} = \psi(x,w)$ with $\psi(x,w) = \tilde{F}_2^{-1}(\phi(x),\hat{F}_2(\phi(x),w))$ and the systems $\Xi_{\mathcal{S}}$ and $\tilde{\Xi}_{\tilde{\mathcal{S}}}$ are feedback equivalent since $\hat{\Xi}_{\tilde{\mathcal{S}}}$ is the image of $\Xi_{\mathcal{S}}$ under the diffeomorphism $\tilde{x} = \phi(x)$.

(ii)⇒(i). Assume that the regular parametrisations $\Xi_{\mathcal{S}} : \dot{x} = F(x,w)$ and $\tilde{\Xi}_{\tilde{\mathcal{S}}} : \dot{\tilde{x}} = \tilde{F}(\tilde{x},\tilde{w})$ of $\mathcal{S}$ and $\tilde{\mathcal{S}}$, respectively, are feedback equivalent via $\tilde{x} = \phi(x)$ and $\tilde{w} = \psi(x,w)$. Denote $\Phi = \phi^{-1}$, and apply the diffeomorphism $x = \Phi(\tilde{x})$ to $\tilde{\Xi}_{\tilde{\mathcal{S}}}$ to obtain a new vector field (parametrised by $\tilde{w}$) $\bar{F}(x,\tilde{w}) = D\Phi(\phi(x))\tilde{F}(\phi(x),\tilde{w})$ related with $F(x,w)$ by the pure feedback transformation $\tilde{w} = \psi(x,w)$. In coordinates $x = (z,y)$, with $y = (y_1,\ldots,y_{n-1})$, we set $F = (F_1,F_2)^t$ and $\bar{F} = (\bar{F}_1,\bar{F}_2)^t$. There is no loss of generality to assume that $\mathrm{rk}\,\frac{\partial F_2}{\partial w}(x_0,w_0) = \mathrm{rk}\,\frac{\partial \bar{F}_2}{\partial \tilde{w}}(x_0,\tilde{w}_0) = n-1$.

Now, we apply to $\tilde{\mathcal{S}}$ the same diffeomorphism $x = \Phi(\tilde{x})$ and denote $\bar{S}(x,\dot{x}) = \tilde{S}(\phi(x), D\phi(x)\dot{x})$. By Lemma 1.1, we conclude that $\frac{\partial S}{\partial \dot{z}} \neq 0$ and that $\frac{\partial \bar{S}}{\partial \dot{z}} \neq 0$. Hence, using the implicit function theorem we get $S(x,\dot{x}) = \delta(x,\dot{x})(\dot{z} - s(x,\dot{y}))$ and $\bar{S}(x,\dot{x}) = \bar{\delta}(x,\dot{x})(\dot{z} - \bar{s}(x,\dot{y}))$, with $\delta \neq 0$ and $\bar{\delta} \neq 0$. By assumption we have, $S(x, F(x,w)) = 0$ and $\bar{S}(x, \bar{F}(x,\tilde{w})) = 0$ implying that $s(x,\dot{y}) = \bar{s}(x,\dot{y})$ (recall that $F_1(x, \psi(x,\tilde{w})) = \bar{F}_1(x,\tilde{w})$ for all w). Hence, we deduce that

$$S(x,\dot{x}) = \delta(x,\dot{x})(\dot{z} - s(x,\dot{y})) = \delta(x,\dot{x})(\dot{z} - \bar{s}(x,\dot{y})) = \frac{\delta}{\bar{\delta}}\tilde{S}(\phi(x), D\phi(x)\dot{x})$$

establishing the equivalence between $\mathcal{S}$ and $\tilde{\mathcal{S}}$.

(ii)⇒*(iii)*. If $\Xi_{\mathcal{S}}$ and $\tilde{\Xi}_{\tilde{\mathcal{S}}}$ are feedback equivalent , then

$$D\phi(x)F(x,w) = \tilde{F}(\phi(x), \psi(x,w)).$$

Thus the diffeomorphism $\varphi(x,w) = (\phi(x), \psi(x,w))$, of the augmented state space (x,w), together with the feedback

$$\tilde{u} = \frac{\partial \psi}{\partial x}F(x,w) + \frac{\partial \psi}{\partial w}u$$

transform $\Sigma_{\mathcal{S}}$ into $\tilde{\Sigma}_{\tilde{\mathcal{S}}}$.

(iii)⇒*(ii)*. Assume that $\Sigma_{\mathcal{S}}$ into $\tilde{\Sigma}_{\tilde{\mathcal{S}}}$ are feedback equivalent via $(\tilde{x},\tilde{w}) = \varphi(x,w)$ and $\tilde{u} = \alpha(x,w) + \beta(x,w)u$. Since φ_* map the distribution span $\left\{\frac{\partial}{\partial w}\right\}$ into span $\left\{\frac{\partial}{\partial \tilde{w}}\right\}$, it follows that φ has the triangular form $(\phi(x), \psi(x,w))$. Therefore, feedback equivalence of the systems $\Xi_{\mathcal{S}}$ and $\tilde{\Xi}_{\tilde{\mathcal{S}}}$ is established via the diffeomorphism $\tilde{x} = \phi(x)$ and the feedback $\tilde{w} = \psi(x,w)$. ∎

Remark. If a parametrisation $\Xi_{\mathcal{S}}$ does not satisfy the regularity condition rk $\frac{\partial F}{\partial w} = n-1$, then the equivalence *(i)*⇔*(ii)* fails to hold. To see that, consider the submanifold $\mathcal{S} \subset T\mathbb{R}^2$ given by the equation $\dot{z} - \dot{y}^2 = 0$. The following parametrisations (the first one corresponds to a first prolongation, i.e. is regular, and the second is not a first prolongation) of $\mathcal{S}$

$$\Xi_{\mathcal{S}} : \begin{cases} \dot{z} &= w^2 \\ \dot{y} &= w \end{cases} \quad \text{and} \quad \tilde{\Xi}_{\mathcal{S}} : \begin{cases} \dot{z} &= \tilde{w}^4 \\ \dot{y} &= \tilde{w}^2 \end{cases}$$

are not feedback equivalent around $w_0 = 0$ and $\tilde{w}_0 = 0$, and the reason is that $\tilde{\Xi}_{\mathcal{S}}$ fails to satisfy $\frac{\partial \tilde{F}}{\partial \tilde{w}}(\tilde{w}_0) \neq 0$ at $\tilde{w}_0 = 0$. ◆

Remark. Notice that the same proof as that of *(ii)*⇔*(iii)* shows that any two nonlinear systems Ξ and $\tilde{\Xi}$ (which need not be regular parametrisations of submanifolds) are feedback equivalent if and only if their prolongations Ξ^p and $\tilde{\Xi}^p$ are control-affine feedback equivalent (see [Jak90]). ◆

Since affine and linear submanifolds, i.e. linear and affine Pfaffian equations, are the most studied ones, their characterisation is known, and, to illustrate our methodology we will give it below with the help of feedback equivalence of first and second prolongations. Observe that since $\mathcal{S}$ is given by a scalar equation in the tangent bundle of a n-dimensional manifold $\mathcal{X}$, then its first prolongation $\Xi_{\mathcal{S}}$ has its state space $\mathcal{X}$ of dimension n and possesses $m = n-1$ controls and its second prolongation $\Sigma_{\mathcal{S}}$ has its state space of dimension $n + m = 2n - 1 = 2m + 1$ and possesses the same number $m = n - 1$ of controls. Recall the distributions $\mathcal{D}^0$ and $\mathcal{D}^1$ defined by (1.5) for any control-affine system $\Sigma = (f, g)$.

Proposition 1.7 (Characterisation of affine and linear Pfaffian equations). *Consider a smooth submanifold $\mathcal{S} \subset T\mathcal{X}$. Then, the following statements are locally equivalent:*

(i) *$\mathcal{S}$ is equivalent to $\mathcal{S}_{aff}$;*

(ii) *Its first prolongation $\Xi_{\mathcal{S}}$ is feedback equivalent to $\Xi_{aff} : \dot{\tilde{x}} = \tilde{f}(\tilde{x}) + \sum_{i=1}^{m} \tilde{g}_i(\tilde{x})\tilde{w}_i$, where $(\tilde{g}_1, \ldots, \tilde{g}_m)$ are linearly independent;*

(iii) *Its second prolongation $\Sigma_{\mathcal{S}}$ is feedback equivalent to*

$$\Sigma_{aff} := \Xi^p_{aff} : \begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{w}} \end{pmatrix} = \begin{pmatrix} \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})\tilde{w} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \mathrm{Id}_m \end{pmatrix} \tilde{u};$$

(iv) *The distributions $\mathcal{D}^0$ and $\mathcal{D}^1$ of the second prolongation $\Sigma_{\mathcal{S}}$ satisfy $[\mathcal{D}^0, \mathcal{D}^1] \subset \mathcal{D}^0$;*

Moreover, the following statements are locally equivalent:

(i)' *$\mathcal{S}$ is equivalent to $\mathcal{S}_{lin}$;*

(ii)' *Its first prolongation $\Xi_{\mathcal{S}}$ is feedback equivalent to $\Xi_{lin} : \dot{\tilde{x}} = \tilde{g}(\tilde{x})\tilde{w}$, where $(\tilde{g}_1, \ldots, \tilde{g}_m)$ are linearly independent;*

(iii)' *Its second prolongation $\Sigma_{\mathcal{S}}$ is feedback equivalent to*

$$\Sigma_{lin} := \Xi^p_{lin} : \begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{w}} \end{pmatrix} = \begin{pmatrix} \tilde{g}(\tilde{x})\tilde{w} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \mathrm{Id}_m \end{pmatrix} \tilde{u};$$

(iv)' *The distributions $\mathcal{D}^0$ and $\mathcal{D}^1$ of the second prolongation $\Sigma_{\mathcal{S}}$ satisfy $[\mathcal{D}^0, \mathcal{D}^1] \subset \mathcal{D}^0$ and, additionally, the drift vector field $f(x,w) = \begin{pmatrix} F(x,w) \\ 0 \end{pmatrix}$ of $\Sigma_{\mathcal{S}}$ satisfies $f(x,0) \in \mathcal{D}^1$;*

Notice that if *(ii)*, *(iii)*, or *(iv)* (resp. *(ii)'*, *(iii)'*, or *(iv)'*) holds for one prolongation (first or second) then it holds for all and it is enough to check the conditions of *(iv)* (resp. *(iv)'*) for just one prolongation.

Proof. The equivalences *(i)*⇔*(ii)*⇔*(iii)* are a direct corollary of Proposition 1.6, since a first prolongation (regular parametrisation) of $\mathcal{S}_{aff}$ is given by a control system of the form Ξ_{aff}, where the vector fields $\tilde{f}$ and $\tilde{g}_i$ can be choosen as follows. Assume that $\mathcal{S}_{aff}$ is given by an equation of the form $\omega(x)\dot{x} + h(x) = 0$, where ω is a nonvanishing differential one-form and h is a smooth function, then choose $\tilde{g}_i$'s such that $\mathrm{span}\,\{\omega\} = \mathrm{ann}\,(\mathrm{span}\,\{\tilde{g}_1, \ldots, \tilde{g}_m\})$ and choose any vector field $\tilde{f}$ satisfying $\langle \omega, \tilde{f} \rangle = -h$. We now prove the equivalence *(iii)*⇔*(iv)*. Necessity of the conditions of *(iv)* is clear from the following explicit form of the distributions $\mathcal{D}^0$ and $\mathcal{D}^1$ of Σ_{aff}:

$$\mathcal{D}^0 = \mathrm{span}\left\{ \frac{\partial}{\partial \tilde{w}_1}, \ldots, \frac{\partial}{\partial \tilde{w}_m} \right\}, \quad \mathcal{D}^1 = \mathcal{D}^0 + \mathrm{span}\left\{ \begin{pmatrix} \tilde{g}_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} \tilde{g}_m \\ 0 \end{pmatrix} \right\}.$$

Recall that, since $\mathcal{D}^0$ is involutive, both $\mathcal{D}^0$ and $\mathcal{D}^1$ are invariant under feedback transformations. Conversely, assume that the distributions

$$\mathcal{D}^0 = \mathrm{span}\left\{ \frac{\partial}{\partial w_1}, \ldots, \frac{\partial}{\partial w_m} \right\}, \quad \text{and} \quad \mathcal{D}^1 = \mathcal{D}^0 + \mathrm{span}\left\{ \begin{pmatrix} \frac{\partial F}{\partial w_1} \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} \frac{\partial F}{\partial w_m} \\ 0 \end{pmatrix} \right\}$$

of $\Sigma_{\mathcal{S}}$ satisfy $[\mathcal{D}^0, \mathcal{D}^1] \subset \mathcal{D}^0$. Therefore, we deduce the condition $\frac{\partial^2 F}{\partial w_i \partial w_j} = 0$ for all $1 \leq i, j \leq m$, meaning that $F(x, w) = f(x) + \sum_{i=1}^{m} g_i(x) w_i$.

The equivalences *(i)'*$\Leftrightarrow$*(ii)'*$\Leftrightarrow$*(iii)'* are again a straightforward corollary of Proposition 1.6. By a similar reasoning as above we show that a first prolongation of $\mathcal{S}_{lin}$ is given by a control system of the form Ξ_{lin}. We now prove *(iii)'*$\Leftrightarrow$*(iv)'*. Necessity of the first two conditions is clear since control-linear systems constitute a subclass of control-affine systems. For Σ_{lin} we have $\tilde{f}(\tilde{x}, \tilde{w}) = \begin{pmatrix} \tilde{g}(\tilde{x})\tilde{w} \\ 0 \end{pmatrix}$, which clearly satisfies $\tilde{f}(\tilde{x}, 0) \in \mathcal{D}^1$, and it is an immediate computation to show that this condition is feedback invariant. Conversely, assume that $\Sigma_{\mathcal{S}}$ satisfies the conditions of *(iv)'*. We immediately conclude that $\Sigma_{\mathcal{S}}$ is feedback equivalent to Σ_{aff}. The remaining condition implies that the vector field $\tilde{f}(\tilde{x})$ can be expressed as a linear combinations of the fields $\tilde{g}_i(\tilde{x})$. Hence, we get

$$\begin{pmatrix} \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})\tilde{w} \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{g}(\tilde{x})(\tilde{\alpha}(\tilde{x}) + \tilde{w}) \\ 0 \end{pmatrix},$$

where the $\tilde{\alpha}_i$'s are smooth functions satisfying $\tilde{f} = \sum_{i=1}^{m} \tilde{\alpha}_i \, \tilde{g}_i$. Introducing a new coordinate $w = \tilde{w} + \alpha$ and using a suitable feedback transformation we get $\dot{w} = u$. Finally, we conclude that (in those coordinates) Σ_{aff} takes the form Σ_{lin}. ■

Remark (Feedback equivalence of control-nonlinear system to control-affine ones). Observe that the equivalence of statements *(ii)* and *(iv)* gives a geometric characterisation of control-nonlinear systems Ξ that are feedback equivalent to a control-affine system Σ (without any prolongation of the state space). ◆

The above proposition gives a characterisation of affine and linear submanifolds, but this is only a beginning of the story. Further investigations are needed to give a classification of those submanifolds, which can also be done under the prism of the classification of their prolongations and has led to a rich literature that we briefly mentioned in the beginning of this section. One may think that second prolongations are needed for classification purposes, but in the case of affine and linear submanifolds their first prolongation is already control-affine (or even control-linear). Therefore it is clearly unnecessary to prolong twice those submanifolds. Moreover, in our methodology we will also consider first prolongations (while they are nonlinear with respect to the control) for the classification problem.

To summarise our methodology, we propose to solve problem 1 and problem 2 via the following equivalent formulations expressed for first and second prolongations of submanifolds.

Problem 1'. Give a characterisation of the feedback equivalence of a control-affine system Σ, with state space a $(2m+1)$-dimensional manifold $\mathcal{M}$ and m controls, with second prolongations of quadric submanifolds $\mathcal{S}_q$.

Problem 2'. Describe some orbits of first prolongation $\Xi_{\mathcal{S}_q}$ of quadric submanifolds under the action of feedback transformations.

3 Contributions and part organisation

In this part of the thesis, we propose a solution of problem 1' and of problem 2', which in turns gives a solution of problems 1 and 2. Our solution covers all nondegenerated quadric submanifolds $\mathcal{S}_q$ when the dimension is $n = 2$, and covers all paraboloid submanifolds $\mathcal{S}_Q$ when the dimension is $n \geq 3$. In the following, we detail for each chapter the problems solved and the results obtained. The results of Chapter 2 have been submitted to the *Journal of Dynamical and Control Systems* [SR21], they also have been presented on several occasions.

Chapter 2: Conic nonholonomic constraints on surfaces and control systems. We develop a complete theory of conic submanifolds (ellipses, hyperbolas, and parabolas) of the tangent bundle of a 2-dimensional manifold $\mathcal{X}$. We give a characterisation of conic submanifolds by a studying the equivalence of second prolongation to a novel type of control-affine systems called quadratic systems. Our characterisation is explicit, that is, it can be tested on any control-affine systems. The conditions that we develop also provide a normal form of conic submanifold which is interesting as it describes a smooth passage from ellipses to hyperbolas through parabolas. Then, the problem of classification of conic submanifolds is considered, we study the problem for each different type of conic separately (elliptic, hyperbolic, and parabolic). Our classification includes several normal and canonical forms.

Chapter 3: Introduction to the equivalence problem of control systems with paraboloid nonholonomic constraints. This chapter is an introduction to the next two, we develop some general tools that are useful for our characterisation and classification of paraboloid submanifolds of any dimension. In particular, we introduce a new class of control-affine systems, called (p, q)-paraboloid system, which are second prolongation of paraboloid submanifolds, and we construct a sort of Hessian map attached to any control-affine system that characterises the signature of the underlying paraboloid submanifold.

Chapter 4: Control systems with paraboloid quadric nonholonomic constraints and Chapter 5: Control systems with paraboloid nonholonomic constraints in any dimension. Those two chapters are dedicated to the study of the problem of characterising and classifying paraboloid submanifolds. First we do it in dimension $n = 3$ (i.e. we have two controls only) and then we generalise our results to the case of arbitrary dimension. In the first case, we study separately elliptic and hyperbolic paraboloid submanifolds, while in the second we embed in one general result our characterisation of paraboloid submanifold. Our characterisation is expressed by relations between the vector fields of control-affine systems and is based on the novel notions of *weak* and *strong* quadratic frames attached to control-affine systems. Our classification results are given for first prolongation of paraboloid submanifold, we give our conditions via relations between structure functions, defined by a moving frame of the tangent bundle, and via geometric invariants. The case of dimension $n = 3$ is therefore a corollary of our result for the arbitrary dimension, but that chapter serves as an introduction for our methodology, and we are also give some interpretations that are not obvious in higher dimensions.

Chapter 6: Characterisation of paraboloid systems by their Lie algebra of infinitesimal symmetries. In this chapter, we use the concept of the Lie algebra of infinitesimal symmetries attached to any control-affine system to characterise a subclass of (p, q)-paraboloid systems. We show that the Lie algebra of infinitesimal symmetries of a second prolongation of a null-form paraboloid submanifold characterises that class of control-affine systems. That methodology gives a direct characterisation of null-forms paraboloid submanifolds among all submanifolds; contrary to our classification results, presented in the previous chapters, that take place inside the class of paraboloid submanifolds.

Chapter 2

Conic nonholonomic constraints on surfaces and control systems

This chapter is based on our paper [SR21] submitted to the *Journal of Dynamical and Control System*. The only notable exceptions are that most of the introduction is now contained in Chapter 1 and that the section dedicated to the study of infinitesimal symmetries of quadratic systems has been moved to Chapter 6.

1 Introduction

In this chapter, we will deal with problems 1 and 2 in the case of dimension $n = 2$. First, we quickly recall the main definitions needed for our study, details are given in Chapter 1. Let $\mathcal{X}$ be a smooth connected manifold of dimension $n = 2$ (a surface), equipped with local coordinates $x = (z, y)$; we choose the order (z, y) to be consistent with some normal forms existing in the literature. In the tangent bundle $T\mathcal{X}$ of $\mathcal{X}$, we consider a smooth 3-dimensional submanifold S given by

$$\mathcal{S} = \{(x, \dot{x}) \in T\mathcal{X},\ S(x, \dot{x}) = 0\},$$

where $S : T\mathcal{X} \to \mathbb{R}$ is a smooth function satisfying $\operatorname{rk} \frac{\partial S}{\partial \dot{x}} = 1$ for all $(x, \dot{x}) \in \mathcal{S}$. Two submanifolds $\mathcal{S} \subset T\mathcal{X}$ and $\tilde{\mathcal{S}} \subset T\tilde{\mathcal{X}}$ given by $S(x, \dot{x}) = 0$ and $\tilde{S}(\tilde{x}, \dot{\tilde{x}}) = 0$, respectively, are said to be equivalent if there exists a diffeomorphism $\phi : \mathcal{X} \to \tilde{\mathcal{X}}$ and a smooth nonvanishing function $\delta : T\mathcal{X} \to \mathbb{R}$ such that

$$\tilde{S}(\phi(x), D\phi(x)\dot{x}) = \delta(x, \dot{x})\, S(x, \dot{x}),$$

where $D\phi$ is the derivative (tangent map) of ϕ.

Example. The submanifolds $\mathcal{S}$ and $\tilde{\mathcal{S}}$ given by $S(x, \dot{x}) = \dot{z} - \left(-1 + \sqrt{1 + \dot{y}}\right)^2 = 0$, around $(x_0, \dot{x}_0) = (0, 0)$, and $\tilde{S}(\tilde{x}, \dot{\tilde{x}}) = \dot{\tilde{z}} - \left(\frac{\dot{\tilde{y}}}{2}\right)^2 = 0$, around $(\tilde{x}_0, \dot{\tilde{x}}_0) = (0, 0)$, respectively, are equivalent via

$$(\tilde{z}, \tilde{y}) = \phi(x) = (z, y - z), \quad \text{and} \quad \delta(x, \dot{x}) = -\frac{1}{4}\left(\dot{z} - \dot{y} - 2 - 2\sqrt{1 + \dot{y}}\right).$$

•

Example. Since on a 2-dimensional manifold all metrics are locally conformally flat then the submanifold given by $S(x,\dot{x}) = a(x)\dot{z}^2 + 2b(x)\dot{z}\dot{y} + c(x)\dot{y}^2 - 1 = 0$ (where $ac - b^2 \neq 0$) is locally equivalent to that given by $\tilde{S}(\tilde{x},\dot{\tilde{x}}) = \left(\dot{\tilde{z}}^2 \pm \dot{\tilde{y}}^2\right) - r(\tilde{x})^2 = 0$. •

The first purpose of this work is to provide a characterisation of submanifolds $\mathcal{S}$ that, at each $x \in \mathcal{X}$, form a conic in $T_x\mathcal{X}$. Namely, in a suitable coordinate system $x = (z,y)$, $\mathcal{S}$ is given by the zero level-set of

$$S_q(x,\dot{x}) = \dot{x}^t \mathsf{g}(x)\dot{x} + 2\omega(x)\dot{x} + h(x).$$

The map S_q is represented by the triple $S_q = (\mathsf{g},\omega,h)$, where g is a smooth symmetric $(0,2)$-tensor (possibly degenerated), ω is a smooth differential one-form and h is a smooth function. It is well-known in affine geometry that quadratic equations can be classified by the signature of the matrix $M_q(x) = \begin{pmatrix} \mathsf{g}(x) & \omega(x)^t \\ \omega(x) & h(x) \end{pmatrix}$ and by that of $\mathsf{g}(x)$. We will use the following two determinants

$$\Delta_1(x) = \det(M_q(x)) \quad \text{and} \quad \Delta_2(x) = \det(\mathsf{g}(x)).$$

Of course, Δ_1 and Δ_2 depend on the choice of coordinates but the ideals generated by them (and thus their zero level set) are invariantly related to the submanifold $\mathcal{S}_q$. In this work, we characterise non-degenerate conics, that is, non-empty and satisfying $\Delta_1 \neq 0$ (observe that degenerate conics are in fact linear subspaces of $T_x\mathcal{X}$). Excluding empty $\mathcal{S}_q$ is needed when considering elliptic submanifold (see the proof of the lemma below) and it implies that M_q is indefinite. The non-degeneracy assumption $\Delta_1 \neq 0$ implies that, locally around x_0, $\operatorname{rk}\mathsf{g}(x) \geq 1$. If $\Delta_2(x_0) \neq 0$ or if $\Delta_2 \equiv 0$ in a neighbourhood of x_0, then we can describe three particular types of conics given by the classification lemma below. Notice, however, that this lemma does not describe conics for which we pass smoothly from one type to another (see remark below the proof of Theorem 2.3 for that case).

Lemma 2.1 (Classification of non-degenerate conics)**.** *Consider a non-empty conic $\mathcal{S}_q$ and assume $\Delta_1 \neq 0$. Then locally around x_0 we have*

(i) *If $\Delta_2(x_0) > 0$, then $\mathcal{S}_q$ is equivalent to $\mathcal{S}_E = \{(\dot{z} - c_0)^2 + (\dot{y} - c_1)^2 = r^2\}$,*

(ii) *If $\Delta_2(x_0) < 0$, then $\mathcal{S}_q$ is equivalent to $\mathcal{S}_H = \{(\dot{z} - c_0)^2 - (\dot{y} - c_1)^2 = r^2\}$,*

(iii) *If $\Delta_2 \equiv 0$, then $\mathcal{S}_q$ is equivalent to $\mathcal{S}_P = \{a\dot{y}^2 - \dot{z} + b\dot{y} + c = 0\}$,*

where r, c_0, c_1, a, b, and c are smooth functions satisfying $r(\cdot) \neq 0$ and $a(\cdot) \neq 0$.

We call $\mathcal{S}_E$, resp. $\mathcal{S}_H$, resp. $\mathcal{S}_P$, an elliptic, resp. a hyperbolic, resp. a parabolic, submanifold and we will use the notation $\mathcal{S}_Q$ to design the set $\{\mathcal{S}_E, \mathcal{S}_H, \mathcal{S}_P\}$ of those three particular forms. Observe that for the parabolic form $\mathcal{S}_P$, the non-degeneracy assumption $\Delta_1 \neq 0$ implies the existence of a nonvanishing differential one-form $\omega = -dz + b\,dy$, whereas in the elliptic and hyperbolic cases this one-form, given by $\omega = -(c_0 dz \pm c_1 dy)$, can vanish at some points. Those three classes of submanifolds are related to the signature of the metric g; indeed if $\operatorname{sgn}(\mathsf{g})$ is constant in a neighbourhood of x_0, then $\mathcal{S}_E$, resp. $\mathcal{S}_H$, resp. $\mathcal{S}_P$, corresponds to $\operatorname{sgn}(\mathsf{g}) = (+,+)$, resp. $\operatorname{sgn}(\mathsf{g}) = (+,-)$, resp. $\operatorname{sgn}(\mathsf{g}) = (+,0)$ (notice that we can always assume that there is at least one positive eigenvalue, otherwise we take the equivalent submanifold given by $\tilde{S} = -S$).

Proof. Consider a submanifold $\mathcal{S}_q$ given in local coordinates by

$$S_q(x,\dot{x}) = \dot{x}^t \begin{pmatrix} \mathsf{g}_{11} & \mathsf{g}_{12} \\ \mathsf{g}_{12} & \mathsf{g}_{22} \end{pmatrix} \dot{x} + 2\,(\omega_1,\ \omega_2)\,\dot{x} + h,$$

where all functions g_{ij}, ω_i, and h depend smoothly on $x \in \mathcal{X}$.

(i)-*(ii)* We deal with $\Delta_2(x_0) \neq 0$, that is, the elliptic and hyperbolic cases together. In those cases, g is a non-degenerate symmetric $(0,2)$-tensor, therefore it can be interpreted as a pseudo-Riemanian metric. Since on 2-dimensional manifolds all metrics are conformally flat, we introduce coordinates $\tilde{x} = \phi(x) = (z,y)$ such that in those coordinates, $\mathcal{S}_q$ can be written (we drop the tildes for more readability),

$$S_q = R^2(x)(\dot{z}^2 \pm \dot{y}^2) + 2\omega(x)\dot{x} + h(x)$$

or, equivalently (since $R(\cdot) \neq 0$), as

$$S_q = R^2\left(\dot{z} + \frac{\omega_1}{R^2}\right)^2 \pm R^2\left(\dot{y} \pm \frac{\omega_2}{R^2}\right)^2 + h - R^2\left(\frac{\omega_1}{R^2}\right)^2 \mp R^2\left(\frac{\omega_2}{R^2}\right)^2.$$

Notice that for this form we have $\Delta_1 = \pm R^2\left(hR^2 - (\omega_1)^2 \mp (\omega_2)^2\right)$ which, by assumption, does not vanish. Denote $c_0 = -\frac{\omega_1}{R^2}$, $c_1 = \mp\frac{\omega_2}{R^2}$, and divide by $\tilde{h} = -h + R^2\left(\frac{\omega_1}{R^2}\right)^2 \pm R^2\left(\frac{\omega_2}{R^2}\right)^2$ (observe that $\tilde{h} \neq 0$ as $\tilde{h} = \frac{1}{R^2}\Delta_1$) to obtain

$$S_q = \frac{R^4}{-\Delta_1}\left[(\dot{z} - c_0)^2 \pm (\dot{y} - c_1)^2\right] - 1. \tag{2.1}$$

In the elliptic case, if $\Delta_1 > 0$ then the conic is empty which is excluded by assumption, therefore we set $r^{-2} = \frac{R^4}{-\Delta_1}$ to obtain $\mathcal{S}_E$. In the hyperbolic case, if $\Delta_1 > 0$, then we permute the variables (z,y) and, thus, we can always obtain a conic defined by S_q in the form (2.1) with $\Delta_1 < 0$. Then, set $r^{-2} = \frac{R^4}{-\Delta_1}$ to obtain $\mathcal{S}_H$.

(iii) Assume $\Delta_2 \equiv 0$. Since $\Delta_1 \neq 0$, we have $\operatorname{rk}\mathsf{g}(x) = 1$ in a neighbourhood, which implies that $\mathsf{g}_{11}(x_0)\mathsf{g}_{22}(x_0) \neq 0$ and thus, without loss of generality, we can assume that $\mathsf{g}_{22}(x_0) \neq 0$. Then, by $\operatorname{rk}\mathsf{g}(x) = 1$, we have $\mathsf{g}_{11} = \frac{(\mathsf{g}_{12})^2}{\mathsf{g}_{22}}$, and the distribution $\ker\mathsf{g} = \operatorname{span}\left\{\mathsf{g}_{22}(x)\frac{\partial}{\partial z} - \mathsf{g}_{12}(x)\frac{\partial}{\partial y}\right\}$ is locally of constant rank. Then we introduce coordinates $(\tilde{z},\tilde{y})$, satisfying $\operatorname{span}\{\mathrm{d}\tilde{y}\} = \operatorname{ann}(\ker\mathsf{g})$, in which we have

$$\tilde{S}_q = \tilde{a}(\tilde{x})\dot{\tilde{y}}^2 + 2\tilde{\omega}(\tilde{x})\dot{\tilde{x}} + \tilde{h}(\tilde{x}),$$

whose determinant $\Delta_1 = -\tilde{a}(\tilde{\omega}_1)^2 \neq 0$ implies that $\tilde{\omega}_1 \neq 0$. Dividing $\tilde{S}_q$ by $-2\tilde{\omega}_1$ we obtain the desired form $\mathcal{S}_P$ with $a = \frac{\tilde{a}}{-2\tilde{\omega}_1}$, $b = \frac{\tilde{\omega}_2}{-\tilde{\omega}_1}$, and $c = \frac{\tilde{h}}{-2\tilde{\omega}_1}$.

■

The second goal of this work is to provide a classification of elliptic, hyperbolic, and parabolic submanifolds. We will give several normalisations and, in particular, we will characterise and propose canonical forms of submanifolds S_q with constant coefficients (called strongly flat in this chapter). Our analysis is based on attaching to a submanifold $\mathcal{S} = \{S(x, \dot{x}) = 0\} \subset T\mathcal{X}$ two control systems. First,

$$\Xi_{\mathcal{S}} \,:\, \dot{x} = F(x, w), \quad x \in \mathcal{X}, \quad w \in \mathcal{W} \subset \mathbb{R},$$

where $\dot{x} - F(x, w) = 0$ is a parametric representation of $\mathcal{S}$, and second,

$$\Sigma_{\mathcal{S}} \,:\, \left\{ \begin{array}{ll} \dot{x} & = F(x, w) \\ \dot{w} & = u \end{array} \right. , \quad (x, w) \in \mathcal{X} \times \mathbb{R}, \quad u \in \mathbb{R},$$

called, respectively, a first and second prolongation of $\mathcal{S}$. An observation that links studying submanifolds $\mathcal{S} \subset T\mathcal{X}$ and their prolongations $\Xi_{\mathcal{S}}$ and $\Sigma_{\mathcal{S}}$ is that equivalence of submanifolds corresponds to the equivalence of control systems $\Xi_{\mathcal{S}}$ and $\Sigma_{\mathcal{S}}$ via feedback transformations, general for $\Xi_{\mathcal{S}}$, and control-affine for $\Sigma_{\mathcal{S}}$, as assured by Proposition 1.6 of Chapter 1.

Example. There is a well known example of a nonlinear system subject to an elliptic constraint $\mathcal{S}_E$, namely Dubin's car [Dub57]. The state of the system is (z, y, w) where (z, y) is the centre of mass of the vehicle, and w is the orientation of the vehicle (with respect to the z-axis). Assume that the vehicle has constant velocity and that we control the angular velocity of the orientation. Thus, the dynamics of Dubin's car reads

$$\left\{ \begin{array}{ll} \dot{z} & = r\cos(w) \\ \dot{y} & = r\sin(w) \\ \dot{w} & = u \end{array} \right. , \quad r \in \mathbb{R}^*,$$

which clearly is a second prolongation of the elliptic submanifold $\dot{z}^2 + \dot{y}^2 = r^2$. •

Related works. A classification of quadratic control systems was initiated by Bonnard in [Bon91]. His work differs from our as he considered homogeneous systems of degree 2 with respect to all state variables. Hence, his class of quadratic control systems is a subclass of our parabolic systems (where we require that only one variable enters quadratically) but he considers the general dimension n while our results concern 3-dimensional systems only. In [KK92], Krener and Kang studied the problem of equivalence, via feedback, to polynomial systems of degree 2 modulo higher order terms. This work was continued in [Kan96] and [TR02] for any degree but all those results are given for formal classification only. Agrachev [Agr98; AG97], see also Agrachev and Sachkov [AS13], has developed a detailed theory of curvature for control-nonlinear systems (compare [JK13] for another approach) and, in particular, described 3-dimensional control-affine systems with zero curvature, see also [Ser09]. It turns out that his models show up in our classification as representatives of null-forms elliptic and hyperbolic systems. Examples of control systems subject to conic nonholonomic constraints appear in various domains of physics and engineering applications. Above we presented Dubin's car [Dub57], which is a simple model of a vehicle, whose evolution is given by a second prolongation of an elliptic submanifold. In [Mon98] an hyperbolic counterpart of Dubin's car have been introduced. We also mention [ZTC15], where the planar tilting manoeuvre problem is considered under small angle assumption, the studied control system is elliptic with respect to the states.

Organisation of the chapter. In section 2, we will define a general second prolongation of a conic submanifold called a quadratic system (see Definition 2.1 and Proposition 2.1) and, in Theorem 2.2, we will fully characterise those systems by means of a checkable relation between well-defined structure functions attached to any control-affine system. The conditions obtained in that theorem allow to give a normal form for all quadratisable control systems, see Theorem 2.3, which in turn leads to a normal form for all conic submanifolds $\mathcal{S}_q$. We will also show how our characterisation and the normal form apply to the class of elliptic, hyperbolic, and parabolic control-affine systems, which gives us a deeper insight into our conditions and in our normal form (see Corollary 2.1 and Corollary 2.2). Finally, in section 3, we will be interested in the classification of elliptic, hyperbolic, and parabolic submanifolds. To every first prolongation of those submanifolds Ξ_E, Ξ_H, or Ξ_P, we attach a frame of the tangent bundle (see the paragraph before Proposition 2.2) and we give conditions for that frame to be commutative: it turns out that in the elliptic and hyperbolic cases this requires that a certain pseudo-Riemanian metric is flat (see Proposition 2.4), whereas in the parabolic case this problem can be solved without any extra assumptions (see Proposition 2.6). Then we show how we can additionally normalise the systems while preserving the commutativity of that frame. Our classification includes several normal and canonical forms, given by Proposition 2.5 for elliptic and hyperbolic systems and by Theorem 2.4 for parabolic systems. Moreover, that classification and those normal and canonical forms, of non-linear quadratic systems, give a corresponding classification and normal/canonical forms of conic submanifolds, see Corollary 2.3 for elliptic and hyperbolic submanifolds $\mathcal{S}_E$ and $\mathcal{S}_H$, and Corollary 2.5 for parabolic submanifolds $\mathcal{S}_P$. Finally, in the case of parabolic systems, we explore the concept of curvature of a control systems defined by Agrachev in [Agr98; AG97] and we characterise and classify all parabolic systems with constant curvature (see Proposition 2.7).

2 Quadratisable control-affine systems

In this section, we address the equivalence problem of a control-affine system Σ to a quadratic control-affine system Σ_q (see definition below), which corresponds to second prolongation of conic submanifolds. On a 3-dimensional manifold $\mathcal{M}$, equipped with local coordinates ξ, we consider the control-affine system

$$\Sigma \,:\, \dot{\xi} = f(\xi) + g(\xi)u,$$

with a scalar control $u \in \mathbb{R}$ and smooth vector fields f and g. For this system, we denote $\mathcal{G} = \operatorname{span}\{g\}$, the distribution spanned by g, and we will use the following notations: given two vector fields g and f on $\mathcal{M}$, by $[g, f]$ we denote the Lie bracket of g and f, in coordinates we have $[g, f] = \frac{\partial f}{\partial \xi} g - \frac{\partial g}{\partial \xi} f$, and $\mathrm{ad}_g^k f = \left[g, \mathrm{ad}_g^{k-1} f\right]$ stands for the iterated Lie bracket, with $\mathrm{ad}_g^0 f = f$, and ϕ_* denotes the tangent map of a diffeomorphism ϕ.

Definition 2.1 (Quadratisable systems)**.** We say that Σ is *quadratisable* if it is feedback equivalent to

$$\Sigma_q \,:\, \begin{cases} \dot{x} &= f_q(x, w) \\ \dot{w} &= u \end{cases},$$

around (x_0,w_0), where f_q satisfies $\left(\frac{\partial^2 f_q}{\partial w^2}\wedge\frac{\partial f_q}{\partial w}\right)(x_0,w_0)\neq 0$ and

$$\frac{\partial^3 f_q}{\partial w^3}=\tau(x)\,\frac{\partial f_q}{\partial w}, \tag{2.2}$$

with τ a smooth function of the indicated variable.

Set $x=(z,y)$ and denote $f_q=f^1\frac{\partial}{\partial z}+f^2\frac{\partial}{\partial y}$; it is shown in Appendix 2.A that any smooth f_q, satisfying (2.2), can be written, locally around (x_0,w_0), as

$$f_q(x,w)=\sum_{k=0}^{+\infty}A(x)\frac{(w-w_0)^{2k+2}}{(2k+2)!}\tau(x)^k+B(x)\frac{(w-w_0)^{2k+1}}{(2k+1)!}\tau(x)^k+C(x), \tag{2.3}$$

where A, B, and C can be seen as smooth vector fields on $\mathcal{X}\cong\mathcal{M}/\mathcal{G}$ for which we have $A\wedge B\neq 0$. The following proposition shows that Σ_q is a second prolongation of a conic submanifold, thus justifies to call Σ_q a quadratic system, and describes three normal forms of f_q given for $\tau\equiv 0$, and $\tau\neq 0$.

Proposition 2.1. *Locally around $\xi_0\in\mathcal{M}$ we have the followings*

(i) *Σ_q is a second prolongation of a conic submanifold $\mathcal{S}_q$.*

(ii) *If $\tau\equiv 0$ in a neighbourhood, resp. $\tau(\xi_0)<0$, resp. $\tau(\xi_0)>0$, then Σ_q is locally feedback equivalent to Σ_P, resp. Σ_E, resp. Σ_H, given by, respectively,*

$$f_P=A(x)w^2+B(x)w+C(x),\qquad f_E=A(x)\cos(\tilde{w})+B(x)\sin(\tilde{w})+C(x),$$
$$f_H=A(x)\cosh(\tilde{w})+B(x)\sinh(\tilde{w})+C(x).$$

(iii) *Σ_P, resp. Σ_E, resp. Σ_H, is a second prolongation of a conic submanifold $\mathcal{S}_Q$ satisfying $\Delta_2\equiv 0$, resp. $\Delta_2>0$, resp. $\Delta_2<0$.*

The conic submanifold $\mathcal{S}_Q$ of statement *(iii)* is, by Lemma 2.1, equivalent to $\mathcal{S}_P$ (if $\Delta_2\equiv 0$), resp. $\mathcal{S}_E$ (if $\Delta_2>0$), resp. $\mathcal{S}_H$ (if $\Delta_2<0$). So it is natural to call Σ_P a parabolic system, Σ_E an elliptic system, and Σ_H a hyperbolic system. We will denote by Q the set $\{P,E,H\}$, and, consequently, $f_Q=\{f_P,f_E,f_H\}$ and $\Sigma_Q=\{\Sigma_P,\Sigma_E,\Sigma_H\}$.

Proof.

(i) Consider f_q given by (2.3), for simplicity we consider $w_0=0$. By Corollary A.1 of Appendix A, we choose coordinates $(z,y)=\phi(x)$ that rectify simultaneously the distributions spanned by A and B so we may assume that $A=a\frac{\partial}{\partial z}$ and $B=b\frac{\partial}{\partial y}$, where a and b are smooth functions of $x=(z,y)$ satisfying $a(x_0)b(x_0)\neq 0$. Using Cauchy products we compute

$$\left(\frac{\dot z-c_0}{a}\right)^2=\left(\sum_{k=0}^{+\infty}\frac{w^{2k+2}}{(2k+2)!}\tau^k\right)^2=\sum_{k=0}^{+\infty}\frac{(8\cdot 4^k-2)w^{2k+4}\,\tau^k}{(2k+4)!},$$

$$\begin{aligned}\left(\frac{\dot y-c_1}{b}\right)^2&=\left(\sum_{k=0}^{+\infty}\frac{w^{2k+1}}{(2k+1)!}\tau^k\right)^2=\sum_{k=0}^{+\infty}\frac{2\cdot 4^k w^{2k+2}\,\tau^k}{(2k+2)!},\\ &=w^2+\sum_{k=1}^{+\infty}\frac{2\cdot 4^k w^{2k+2}\,\tau^k}{(2k+2)!}=w^2+\sum_{k=0}^{+\infty}\frac{2\cdot 4^{k+1}w^{2k+4}\,\tau^{k+1}}{(2k+4)!},\\ &=w^2+\tau\sum_{k=0}^{+\infty}\frac{8\cdot 4^k w^{2k+4}\,\tau^k}{(2k+4)!}=w^2+\tau\left(\frac{\dot z-c_0}{a}\right)^2+\tau\sum_{k=0}^{+\infty}\frac{2w^{2k+4}\,\tau^k}{(2k+4)!}.\end{aligned}$$

Consequently, we obtain

$$\left(\frac{\dot y - c_1}{b}\right)^2 - \tau\left(\frac{\dot z - c_0}{a}\right)^2 = w^2 + \sum_{k=0}^{+\infty}\frac{2w^{2k+4}\,\tau^{k+1}}{(2k+4)!} = \sum_{k=-1}^{+\infty}\frac{2w^{2k+4}\,\tau^{k+1}}{(2k+4)!},$$
$$= \sum_{k=0}^{+\infty}\frac{2w^{2k+2}\,\tau^{k}}{(2k+2)!} = 2\left(\frac{\dot z - c_0}{a}\right).$$

Hence, the Σ_q is a second prolongation of the submanifold $\mathcal{S}_q$ given by $\left(\frac{\dot y - c_1}{b}\right)^2 - \tau\left(\frac{\dot z - c_0}{a}\right)^2 - 2\left(\frac{\dot z - c_0}{a}\right) = 0$ and for which we have $\Delta_1 = -\frac{1}{a^2b^2}$ and $\Delta_2 = -\frac{\tau}{a^2b^2}$.

(ii) If $\tau \equiv 0$, then $\frac{\partial^3 f_q}{\partial w^3} \equiv 0$ implies that $f_q = f_P = Aw^2 + Bw + C$. If $\tau < 0$, then $\frac{\partial^3 f_q}{\partial w^3} = \tau\frac{\partial f_q}{\partial w}$ implies that $f_q = A\cos(\sqrt{-\tau}w) + B\sin(\sqrt{-\tau}w) + C$ which, via the pure feedback $\tilde w = \sqrt{-\tau}w$, gives $f_q = f_E$. Finally, if $\tau > 0$, then $\frac{\partial^3 f_q}{\partial w^3} = \tau\frac{\partial f_q}{\partial w}$ implies that $f_q = A\cosh(\sqrt{\tau}w) + B\sinh(\sqrt{\tau}w) + C$ which, via the pure feedback $\tilde w = \sqrt{\tau}w$, gives $f_q = f_H$.

(iii) To prove that Σ_Q is a second prolongation of a conic submanifold $\mathcal{S}_Q$ satisfying $\Delta_2 \equiv 0$, or $\Delta_2 > 0$, or $\Delta_2 > 0$, consider the distributions $\mathcal{A} = \operatorname{span}\{A\}$ and $\mathcal{B} = \operatorname{span}\{B\}$; under the assumption $A \wedge B \neq 0$, there exists coordinates $(\tilde z, \tilde y)$ such that $\operatorname{span}\{d\tilde y\} = \operatorname{ann}(\mathcal{A})$ and $\operatorname{span}\{d\tilde z\} = \operatorname{ann}(\mathcal{B})$ (see Corollary A.1 of Appendix A). In those coordinates, we have $A = a\frac{\partial}{\partial \tilde z}$ and $B = b\frac{\partial}{\partial \tilde y}$ with $ab \neq 0$. Denote $C = c_0\frac{\partial}{\partial \tilde z} + c_1\frac{\partial}{\partial \tilde y}$. Thus we have,

Σ_P: using the equation for $\dot{\tilde y}$ we obtain $w = \frac{\dot{\tilde y} - c_1}{b}$, which inserted into the equation for $\dot{\tilde z}$, gives $\dot{\tilde z} - a\left(\frac{\dot{\tilde y} - c_1}{b}\right)^2 + c_0 = 0$ for which we clearly have $\Delta_2 \equiv 0$.

Σ_E: we easily see that $\cos(\tilde w) = \frac{\dot{\tilde z} - c_0}{a}$ and $\sin(\tilde w) = \frac{\dot{\tilde y} - c_1}{b}$ so we obtain the conic submanifold $\left(\frac{\dot{\tilde z} - c_0}{a}\right)^2 + \left(\frac{\dot{\tilde y} - c_1}{b}\right)^2 = 1$, which satisfies $\Delta_2 = \frac{1}{a^2b^2} > 0$.

Σ_H: similarly to the elliptic case, we have $\cosh(\tilde w) = \frac{\dot{\tilde z} - c_0}{a}$ and $\sinh(\tilde w) = \frac{\dot{\tilde y} - c_1}{b}$, and thus $\left(\frac{\dot{\tilde z} - c_0}{a}\right)^2 - \left(\frac{\dot{\tilde y} - c_1}{b}\right)^2 = 1$, for which we have $\Delta_2 = \frac{-1}{a^2b^2} < 0$.

■

Notice that τ plays for Σ_q a similar role as the one that Δ_2 plays for $\mathcal{S}_q$ and, indeed, we see in the proof of the first statement that in a suitable coordinate system we have $\Delta_2 = -\tau$.

The remaining part of this section is organised as follows. First, we will state our main theorem giving necessary and sufficient conditions characterising quadratisable systems Σ_q. Second, by carefully studying the conditions of that theorem, we will give a normal form of the quadratisable systems.

2.1 Characterisation of quadratisable control-affine systems

We now focus on the equivalence of a general control-affine system $\Sigma : \dot\xi = f(\xi) + g(\xi)u$ with quadratic control-affine systems Σ_q. The theorem below gives checkable necessary and sufficient conditions in terms of relations between the vector fields f and g of Σ for the existence of a smooth feedback (α, β) and a diffeomorphism $\tilde{\xi} = \phi(\xi)$ that locally transform Σ into a quadratic system Σ_q. Equivalence to particular cases of Σ_q, namely Σ_E, Σ_H, and Σ_P, is provided by Corollary 2.1 below.

Theorem 2.2 (Feedback quadratisation)**.** *Let Σ be a control-affine system on a 3-dimensional smooth manifold with a scalar control. Σ is, locally around $\xi_0 \in \mathcal{M}$, feedback equivalent to a quadratic system Σ_q if and only if*

(C1) $g \wedge \mathrm{ad}_g f \wedge \mathrm{ad}_g^2 f \; (\xi_0) \neq 0$,

(C2) *The structure functions ρ and τ in the decomposition* $\mathrm{ad}_g^3 f = \rho \mathrm{ad}_g^2 f + \tau \mathrm{ad}_g f$ mod $\mathcal{G}$ *satisfy, locally around ξ_0,*

$$\mathrm{L}_g(\chi) - \frac{2}{3}\rho\chi = 0, \tag{2.4}$$

where $\chi = 3\mathrm{L}_g(\rho) - 2\rho^2 - 9\tau$.

Condition (C1) is a regularity condition, it ensures that the vector fields $g, \mathrm{ad}_g f$, and $\mathrm{ad}_g^2 f$ are locally linearly independent and thus that they form a local frame, hence the structure functions (ρ, τ) of (C2) are well defined. The main idea behind this theorem is to observe that if for Σ we have $\mathrm{ad}_g^3 f = \tau(x)\,\mathrm{ad}_g f$, i.e. the third Lie derivative of f along g is proportional to the first Lie derivative of f along g, then with the help of a diffeomorphism we can obtain the form Σ_q, see the sufficiency part of the proof for details. Thus condition (C2) shows how that relation changes when we allow for feedback transformations (α, β).

Proof. **Necessity.** Consider a control-affine system Σ given by two smooth vector fields f and g and recall that $\mathcal{G}$ is the distribution $\mathcal{G} = \mathrm{span}\{g\}$. Let (ϕ, α, β) form a feedback such that Σ is (locally) equivalent to a quadratic system Σ_q. We suppose that $\Sigma_q = (\tilde{f}_q, \tilde{g})$ is given in local coordinates $\tilde{\xi} = (\tilde{z}, \tilde{y}, \tilde{w})$, we set $\tilde{\mathcal{G}}$ the distribution spanned by $\tilde{g}$, and we denote $(\tilde{\rho}, \tilde{\tau})$ its structure functions. By definition of feedback equivalence the following relations between (f, g) and $(\tilde{f}_q, \tilde{g})$ hold: $\tilde{f}_q = \phi_*(f + \alpha g)$ and $\tilde{g} = \phi_*(g\beta)$.
The system Σ_q is quadratic, so by Definition 2.1, we have $\frac{\partial^2 \tilde{f}_q}{\partial \tilde{w}^2} \wedge \frac{\partial \tilde{f}_q}{\partial \tilde{w}}(\tilde{x}_0, \tilde{w}_0) \neq 0$, which implies that (C1) holds for $(\tilde{f}_q, \tilde{g})$, and we also have $\frac{\partial^3 \tilde{f}_q}{\partial \tilde{w}^3} = \tilde{\tau}\frac{\partial \tilde{f}_q}{\partial \tilde{w}}$, thus the structure functions of Σ_q are $\tilde{\rho} = 0$ and $\tilde{\tau} = \tilde{\tau}(\tilde{z}, \tilde{y})$. Therefore, for Σ_q we have $\tilde{\chi} = -9\tilde{\tau}(\tilde{z}, \tilde{y})$ implying $\mathrm{L}_{\tilde{g}}(\tilde{\chi}) = \frac{\partial \tilde{\chi}}{\partial \tilde{w}} = 0$. Hence Σ_q satisfies (C1) and (C2) and we will now prove that those conditions are invariant under diffeomorphisms ϕ and feedback transformations (α, β).
Clearly, (C1) is invariant under diffeomorphisms (as $[\phi_* g, \phi_* f] = \phi_*[g, f]$) and under feedback (α, β) since $\beta \neq 0$. We have checked that (C2) holds for $\Sigma_q = (\tilde{f}_q, \tilde{g})$ and, clearly, (C2) is invariant under diffeomorphisms since they conjugate structure functions. Moreover (C2) is invariant under the transformation $\tilde{f}_q \mapsto \tilde{f}_q + \alpha\tilde{g}$, since the expression of $\mathrm{ad}_{\tilde{g}}^3 \tilde{f}_q$ is considered modulo the distribution $\tilde{\mathcal{G}}$. Finally, under the

action of β the brackets, with $\tilde{g} = g\beta$, are transformed by,

$$
\begin{aligned}
\mathrm{ad}_{\tilde{g}} f &= \beta \mathrm{ad}_g f \mod \tilde{\mathcal{G}}, \\
\mathrm{ad}^2_{\tilde{g}} f &= \beta^2 \mathrm{ad}^2_g f + \beta \left(\mathrm{L}_g\left(\beta\right)\right) \mathrm{ad}_g f \mod \tilde{\mathcal{G}}, \\
\mathrm{ad}^3_{\tilde{g}} f &= (\beta^3 \rho + 3\beta^2 \mathrm{L}_g\left(\beta\right)) \mathrm{ad}^2_g f + (\beta^3 \tau + \beta \mathrm{L}_g\left(\beta \mathrm{L}_g\left(\beta\right)\right)) \mathrm{ad}_g f \mod \tilde{\mathcal{G}}, \\
&= (\rho\beta + 3\mathrm{L}_g\left(\beta\right)) \mathrm{ad}^2_{\tilde{g}} f \\
&\quad + \left(\tau\beta^2 + \mathrm{L}_g\left(\beta \mathrm{L}_g\left(\beta\right)\right) - \rho\beta \mathrm{L}_g\left(\beta\right) - 3\left(\mathrm{L}_g\left(\beta\right)\right)^2\right) \mathrm{ad}_{\tilde{g}} f \mod \tilde{\mathcal{G}}.
\end{aligned}
$$

This implies that the structure functions $\tilde{\rho}$ and $\tilde{\tau}$ of Σ_q defined by $\mathrm{ad}^3_{\tilde{g}} f = \tilde{\rho} \mathrm{ad}^2_{\tilde{g}} f + \tilde{\tau} \mathrm{ad}_{\tilde{g}} f \mod \tilde{\mathcal{G}}$ are given in terms of the feedback transformation β and the structure functions ρ and τ of Σ by

$$
\tilde{\rho} = \rho\beta + 3\mathrm{L}_g\left(\beta\right), \quad \tilde{\tau} = \tau\beta^2 + \mathrm{L}_g\left(\beta \mathrm{L}_g\left(\beta\right)\right) - \rho\beta \mathrm{L}_g\left(\beta\right) - 3\left(\mathrm{L}_g\left(\beta\right)\right)^2. \tag{2.5}
$$

Since for Σ_q the structure function $\tilde{\rho}$ vanishes, $\tilde{\rho} = 0$, we have the relation $\mathrm{L}_g\left(\beta\right) = -\frac{\beta\rho}{3}$ and thus $\tilde{\chi} = -9\tilde{\tau}$, which is equal to

$$
\begin{aligned}
\tilde{\chi} = -9\left(\tau\beta^2 + \mathrm{L}_g\left(\beta \mathrm{L}_g\left(\beta\right)\right)\right) &= -9\left(\tau\beta^2 + \mathrm{L}_g\left(\frac{-\beta^2\rho}{3}\right)\right), \\
&= -9\left(\tau\beta^2 - \frac{1}{3}\left(\rho \mathrm{L}_g\left(\beta^2\right) + \beta^2 \mathrm{L}_g\left(\rho\right)\right)\right) \\
&= -9\beta^2\left(\tau - \frac{1}{3}\mathrm{L}_g\left(\rho\right) + \frac{2}{9}\rho^2\right) = \beta^2\chi.
\end{aligned}
$$

And finally, using that $\mathrm{L}_{\tilde{g}}\left(\tilde{\chi}\right) = 0$, we deduce that

$$
\mathrm{L}_{\tilde{g}}\left(\tilde{\chi}\right) = \beta \mathrm{L}_g\left(\beta^2\chi\right) = \beta^3 \mathrm{L}_g\left(\chi\right) + 2\beta^2\chi \mathrm{L}_g\left(\beta\right) = \beta^3 \mathrm{L}_g\left(\chi\right) - \frac{2}{3}\beta^3\chi\rho = 0,
$$

showing the necessity of relation (2.4) and concludes the necessity part of the proof.

Sufficiency. There are two steps in the sufficiency part. The first one consists in building a vector field g such that $\mathrm{ad}^3_g f = \tau \, \mathrm{ad}_g f \mod \mathcal{G}$ with $\mathrm{L}_g\left(\tau\right) = 0$. Then we will construct a diffeomorphism ϕ that brings Σ into the form Σ_q.
Consider the system $\Sigma : \dot{\xi} = f + gu$, for which we assume $g \wedge \mathrm{ad}_g f \wedge \mathrm{ad}^2_g f\left(\xi_0\right) \neq 0$ and suppose that relation (2.4) holds for its structure functions ρ and τ. Choose a function $\beta \neq 0$ satisfying $\mathrm{L}_g\left(\beta\right) = \frac{-\beta\rho}{3}$ (to guarantee that $\beta \neq 0$, we actually may solve the equation $\mathrm{L}_g\left(\ln(\beta)\right) = -\frac{\rho}{3}$). Define the system $\tilde{\Sigma} : \dot{\xi} = \tilde{f} + \tilde{g}\tilde{u}$, where $\tilde{g} = g\beta$ and $\tilde{f} = f$, then by (2.5) the structure function $\tilde{\rho}$ of $\tilde{\Sigma}$ vanishes. Therefore, we have $\tilde{\chi} = -9\tilde{\tau}$ and thus by relation (2.4) we obtain that $\mathrm{L}_{\tilde{g}}\left(\tilde{\tau}\right) = 0$.

Since $\tilde{g} \neq 0$, we apply a diffeomorphism $(z, y, w) = \phi(\xi)$ such that $\phi_* \tilde{g} = g = \frac{\partial}{\partial w}$ and denote $f = \phi_* \tilde{f}$, and $\tau = \tilde{\tau} \circ \phi^{-1}$. Therefore, the decomposition $\mathrm{ad}^3_g f = \tau \mathrm{ad}_g f \mod \mathcal{G}$ implies that $f = f^1 \frac{\partial}{\partial z} + f^2 \frac{\partial}{\partial y} + f^3 \frac{\partial}{\partial w}$ satisfies

$$
\frac{\partial^3 f^i}{\partial w^3} = \tau(z, y) \frac{\partial f^i}{\partial w},
$$

for $i = 1, 2$. Applying the feedback $u = f^3(z, y, w) + \tilde{u}$ we obtain the form Σ_q. ■

The following corollary shows that we can test on the structure functions of Σ if the equivalent quadratic system Σ_q will be of parabolic, elliptic, or hyperbolic type.

Corollary 2.1. *Under conditions (C1) and (C2) of the previous theorem we have*

(i) *Σ is locally feedback equivalent to Σ_P if and only if $\chi \equiv 0$ in a neighbourhood of ξ_0,*

(ii) *Σ is locally feedback equivalent to Σ_E if and only if $\chi(\xi_0) > 0$,*

(iii) *Σ is locally feedback equivalent to Σ_H if and only if $\chi(\xi_0) < 0$,*

Notice that Σ is locally feedback equivalent to Σ_P if and only if it satisfies (C1) and $\chi \equiv 0$, condition (C2) being satisfied automatically.

Proof. From the necessity part of the proof of Theorem 2.2 we know that for Σ_q we have $\tilde{\chi} = -9\tilde{\tau}$ and we saw that under pure feedback transformations (α, β) we have $\tilde{\chi} = \beta^2\chi$, thus the sign of χ is invariant as well as the locus where it vanishes. Moreover, Σ_Q is parabolic if $\tilde{\tau} \equiv 0$, equivalently $\tilde{\chi} \equiv 0$, Σ_Q is equivalent to Σ_E if $\tilde{\tau} > 0$, equivalently $\tilde{\chi} < 0$, and Σ is equivalent to Σ_H if $\tilde{\tau} > 0$, equivalently $\tilde{\chi} < 0$. Hence the necessity of the stated conditions is established.

Conversely, in the sufficiency part of the proof of Theorem 2.2 we obtained structure functions $(\tilde{\rho}, \tilde{\tau}) = (0, \tilde{\tau}(z, y))$ by feedback transformations. Since $\tilde{\chi} = \beta^2\chi$ we have $-9\tilde{\tau} = \beta^2\chi$, thus $\mathrm{sgn}\,(\tilde{\tau}) = -\mathrm{sgn}\,(\chi)$ and the conclusion follows by Proposition 2.1. ∎

2.2 Normal form of quadratisable control-affine system

In this subsection, we apply the conditions of Theorem 2.2 to a general control-affine system, which under the regularity assumption $g \wedge \mathrm{ad}_g f(\xi_0) \neq 0$ can be written (after applying a suitable feedback transformation) as

$$\Sigma_h \; : \; \begin{cases} \dot{z} &= h(z, y, w) \\ \dot{y} &= w \\ \dot{w} &= u \end{cases},$$

with h a smooth function. By applying Theorem 2.2 we are able to give a normal form of all smooth functions $h(z, y, w)$ that describe quadratisable systems, that is, systems feedback equivalent to Σ_q. In what follows, we assume to work locally around $0 \in \mathbb{R}^3$ and all derivatives are taken with respect to w and denoted by prime, double prime, etc. Whenever we apply $\ln(a)$, we assume that $a > 0$ (if not, take the absolute value).

Theorem 2.3 (Normal form of quadratisable control-affine systems)**.** *The following statements are equivalent.*

(i) *Σ_h is locally feedback equivalent to a quadratic system Σ_q;*

(ii) *The function h satisfies $h''(0) \neq 0$ and, in a neighbourhood, it holds*

$$9h''''' \left(h''\right)^2 - 45h''''h'''h'' + 40 \left(h'''\right)^3 = 0; \tag{2.6}$$

(iii) The second derivative of h is of the following form

$$h''(x,w) = a(dw^2 + ew + 1)^{-3/2}, \tag{2.7}$$

where $a = a(x)$, $d = d(x)$, and $e = e(x)$ are smooth functions satisfying $a(0) \neq 0$;

(iv) The function h is given by

$$h(x,w) = 2a\left(\frac{w^2}{(\sqrt{dw^2 + ew + 1} + 1)^2 - dw^2}\right) + bw + c \tag{2.8}$$

where a, b, c, d, e are any smooth function of x such that $a(0) \neq 0$.

Proof. (i)⇔(ii). It is a straightforward application of the conditions of Theorem 2.2 with the structure functions of Σ_h given by $\rho = \frac{h'''}{h''}$ and $\tau = 0$. Then condition (C2) reads

$$\chi' - \frac{2}{3}\rho\chi = 3\rho'' - 6\rho\rho' + \frac{4}{3}\rho^3 = 0, \tag{2.9}$$

which can be expended into (2.6).
To prove *(ii)⇔(iii)*, we will solve (2.9) for ρ and then integrate $\rho = (\ln(h''))'$ to obtain the form (2.7). By a change of variable, it is easy to obtain that the solutions of (2.9) are of the following form (see Section 2.B.1 for the proof)

$$\rho(x,w) = -\frac{3}{2}\frac{2d(x)w + e(x)}{d(x)w^2 + e(x)w + 1}. \tag{2.10}$$

This form can be integrated into $h''(x,w) = a(x)\left(d(x)w^2 + e(x)w + 1\right)^{-3/2}$ with a, d, and e any smooth functions such that $a(0) \neq 0$.
(iii)⇔(iv). To show (2.8), we integrate twice the second derivative of h given by (2.7). We set $p = p(x,w) = d(x)w^2 + e(x)w + 1$ and $\Delta = \Delta(x) = e(x)^2 - 4d(x)$. First, we obtain (see Section 2.B.2 for details)

$$h'(x,w) = \frac{2aw\left(\sqrt{p} + 1\right)}{\sqrt{p}(ew + 2 + 2\sqrt{p})} + b,$$

with b an arbitrary smooth function of x. Integrate once more to get h,

$$\begin{aligned} h(x,w) &= \frac{2a}{\Delta\sqrt{p}}(ew\sqrt{p} - 2p) + \frac{4a}{\Delta} + bw + c = \frac{2a}{\Delta}(ew + 2 - 2\sqrt{p}) + bw + c, \\ &= \frac{2aw^2}{ew + 2 + 2\sqrt{p}} + bw + c = \frac{2aw^2}{(\sqrt{p} + 1)^2 - dw^2} + bw + c. \end{aligned}$$

(iv)⇔(i). Given Σ_h with h defined by (2.8) we construct a feedback transformation that brings the system into Σ_q. First, using Corollary A.1 of Appendix A, we introduce coordinates $(\tilde{z}, \tilde{y}) = \phi(z,y)$ such that $\phi_* \frac{\partial}{\partial z} = \tilde{a}\frac{\partial}{\partial \tilde{z}}$ and $\phi_*\left(b\frac{\partial}{\partial z} + \frac{\partial}{\partial y}\right) = \tilde{b}\frac{\partial}{\partial \tilde{y}}$ and, second, we set $\tilde{w}^2 = \frac{w^2}{ew+2+2\sqrt{p}}$ or, equivalently, $w = \tilde{w}\left(e\tilde{w} \pm 2\sqrt{d\tilde{w}^2 + 1}\right)$. Those transformations bring Σ into the system

$$\left\{\begin{array}{ll} \dot{\tilde{z}} &= 2\tilde{a}\tilde{w}^2 + \tilde{c} \\ \dot{\tilde{y}} &= \tilde{e}\tilde{w}^2 \pm 2\tilde{w}\sqrt{\tilde{d}\tilde{w}^2 + 1} \\ \dot{\tilde{w}} &= u \end{array}\right. . \tag{2.11}$$

The structure functions of the above system are given by $\tilde{\rho} = \frac{-3\tilde{w}\tilde{d}}{\tilde{d}\tilde{w}^2+1}$ and $\tilde{\tau} = \frac{3\tilde{d}}{\tilde{d}\tilde{w}^2+1}$, then apply the feedback $\beta(\tilde{x}, \tilde{w}) = \sqrt{\tilde{d}\tilde{w}^2 + 1}$ (it is a solution of the equation $\frac{\partial \beta}{\partial \tilde{w}} = -\frac{\rho\beta}{3}$) to obtain a new vector field $\bar{g} = \beta \frac{\partial}{\partial \tilde{w}}$ and structure functions are given by $\bar{\rho} = 0$ and $\bar{\tau} = 4\tilde{d}(x)$. To complete the form, it remains to find a new $\bar{w} = \psi(\tilde{x}, \tilde{w})$ satisfying $\psi_* \beta \frac{\partial}{\partial \tilde{w}} = \frac{\partial}{\partial \bar{w}}$. In general ψ is given in terms of the following integral

$$\bar{w} = \psi(\tilde{x}, \tilde{w}) = \int_0^{\tilde{w}} \frac{1}{\sqrt{1 + \tilde{d}\tilde{w}^2}} \, d\tilde{w}.$$

■

Remark. This theorem provides a normal form of submanifolds $\mathcal{S} = \{\dot{z} = s(x, \dot{y})\}$ that are equivalent to a conic submanifold $\mathcal{S}_q$, namely, they are represented by $s(x, \dot{y}) = h(x, \dot{y})$ with h as in (2.8). Moreover, system (2.11) leads to a normal form for all conic submanifold $\mathcal{S}_q$ (even for those that smoothly pass through $\Delta_2(x) = 0$) and thus completes Lemma 2.1. Indeed, for (2.11) we have (tildes have been removed for more readability) $S_q = (e(\dot{z} - c) - 2a\dot{y})^2 - 8a(\dot{z} - c)(\frac{d}{2a}(\dot{z} - c) + 1)$, for which we can compute $\Delta_1 = -64a^4 \neq 0$ and $\Delta_2 = -16a^2 d$. Hence, if $d(0) = 0$ and $d(x) \neq 0$ elsewhere, then $\mathcal{S}_q$ goes smoothly from an elliptic to a hyperbolic submanifold by passing (at $x = 0$) by a parabolic one. ◆

In the last item of the above proof, we saw that the function d plays an import role for the shape of the transformation, in the following corollary we show that this function is the key of the normal form of quadratisable control systems.

Corollary 2.2. *Assume that Σ_h is given by h of the form (2.8). Then Σ is feedback equivalent to Σ_P, resp. Σ_E, resp. Σ_H if and only if $d \equiv 0$, resp. $d < 0$, resp. $d > 0$. Moreover, the normalising feedback transformation is given by*

$$\bar{w} = \frac{w}{1 + \sqrt{ew + 1}} \quad \textit{for} \quad \Sigma_P, \quad \textit{resp.} \quad \sin^2(\sqrt{-d}\bar{w}) = \frac{-dw^2}{ew + 2 + 2\sqrt{p}} \quad \textit{for} \quad \Sigma_E,$$

$$\textit{resp.} \quad \sinh^2(\sqrt{d}\bar{w}) = \frac{dw^2}{ew + 2 + 2\sqrt{p}} \quad \textit{for} \quad \Sigma_H,$$

where $p = p(x, w) = d(x)w^2 + e(x)w + 1$.

Proof. First we show that Σ_h is feedback equivalent to Σ_P, resp. Σ_E, resp. Σ_H, if and only if $d \equiv 0$, resp. $d < 0$, and resp. $d > 0$. From Corollary 2.1 we know that we have to compute the sign of χ, which is given by $\chi = -9\frac{d}{p}$ for Σ_h (this can easily be deduced from the expression of ρ given by (2.10)). Since $p(0) = 1 > 0$ we locally have $\operatorname{sgn}(\chi) = -\operatorname{sgn}(d)$ and thus the conclusion follows.
We now show how to transform Σ_h into Σ_P, resp. Σ_E, resp. Σ_H. From the last part of the proof of the previous theorem we know that a suitable parametrisation $\bar{w}$ is given by the following two steps

$$\tilde{w}^2 = \frac{w^2}{ew + 2 + 2\sqrt{p}}, \quad \text{and} \quad \bar{w} = \int_0^{\tilde{w}} \frac{1}{\sqrt{1 + d\tilde{w}^2}} \, d\tilde{w}.$$

Assume $d \equiv 0$, then the procedure reduces to the first step only and thus $\bar{w}^2 = \tilde{w}^2 = \frac{w^2}{ew+2+2\sqrt{ew+1}} = \left(\frac{w}{1+\sqrt{ew+1}}\right)^2$ and thus we choose $\tilde{w} = \frac{w}{1+\sqrt{ew+1}}$. Assume $d < 0$, then the second step of the procedure leads to $\bar{w} = \frac{1}{\sqrt{-d}}\operatorname{arcsin}(\sqrt{-d}\tilde{w})$. Hence a reparametrisation is given by $\sin^2(\sqrt{-d}\bar{w}) = \frac{-dw^2}{ew+2+2\sqrt{p}}$. Assume $d > 0$, then from the second step of the procedure we have $\bar{w} = \frac{1}{\sqrt{d}}\operatorname{arcsinh}(\sqrt{d}\tilde{w})$. Hence a reparametrisation is given by $\sinh^2(\sqrt{d}\bar{w}) = \frac{dw^2}{ew+2+2\sqrt{p}}$. ■

Remark (Interpretation of parametrising functions). In the normal form (2.8), there are 5 parametrising functions*. However, only $d = d(x)$ and $e = e(x)$ play a significant role in the shape of the submanifold $\mathcal{S}_q$. Indeed, a is a scaling of the submanifold, c is the value of h at $w = 0$, and by an appropriate choice of coordinates we can always assume that $b \equiv 0$. From the above corollary, the role of d is clear: its sign around $x = 0$ determines the nature of the submanifold, that is, whether the submanifold is elliptic, hyperbolic, or parabolic.

The role of the function e is, however, more subtle. Clearly, h is well defined whenever $p > 0$ and, for a given d, the function e determines the region in which $p > 0$ (in particular, whether h is defined globally with respect to w or not). If $d \equiv 0$, then $p > 0$ holds everywhere (i.e. h is defined globally) if and only if $e \equiv 0$ that is, h is explicitly given by $h = 2aw^2 + bw + c$. If $d < 0$, then we have $p > 0$ only between its roots and the parametrisation is never global. Finally, if $d > 0$ then the parametrisation is global if and only if $\Delta < 0$ (where Δ is the discriminant of p), that is $|e| < 2\sqrt{d}$.

◆

3 Classification of quadratic systems

From Theorem 2.2 we know how to characterise control-affine systems equivalent to the quadratic form Σ_q and, in particular, we know how to characterise the subclasses Σ_E, Σ_H, and Σ_P (see Corollary 2.1). We are now interested in classifying, under feedback action, the systems inside those three subclasses. To this end, we consider the quadratic nonlinear system

$$\Xi_Q \,:\, \dot{x} = f_Q(x, w),$$

where $x \in \mathcal{X}$ is the state, $w \in \mathbb{R}$ plays the role of a control that enters in a nonlinear way, and f_Q is given by either

$$\begin{aligned} f_E &= A(x)\cos(w) + B(x)\sin(w) + C(x), \quad \text{or} \\ f_H &= A(x)\cosh(w) + B(x)\sinh(w) + C(x), \quad \text{or} \\ f_P &= A(x)w^2 + B(x)w + C(x). \end{aligned}$$

In each of the three cases, A, B, and C are smooth vector fields on $\mathcal{X}$ satisfying $(A \wedge B)(x_0) \neq 0$. A quadratic nonlinear system Ξ_Q is then represented by the triple (A, B, C) of three smooth vector fields satisfying $A \wedge B \neq 0$. We call the pair (A, B) a *Q-frame*, and if, additionally, $[A, B] = 0$, then we call (A, B) a *commutative Q-frame*.

The following proposition gives the class of admissible reparametrisations (i.e. pure feedback transformations that preserve quadratic systems Ξ_Q) for each quadratic class and shows how those transformations act on the triple (A, B, C).

*On https://www.geogebra.org/m/tyb4ygpb the reader can play with those parameters (the functions a, b, c, d, and e become real numbers when fixing $x \in \mathcal{X}$).

Proposition 2.2 (Reparametrisation of quadratic nonlinear systems)**.**

(i) *Two elliptic systems Ξ_E and $\tilde{\Xi}_E$ are feedback equivalent if and only if there exists a diffeomorphism $\tilde{x} = \phi(x)$ and an invertible reparametrisation (nonlinear feedback) $w = \psi(x, \tilde{w})$, of the form $\psi = \pm\tilde{w} + \alpha(x)$, and satisfying*

$$\tilde{A} = \phi_* \left(A \cos\alpha + B \sin\alpha\right), \quad \tilde{B} = \pm\phi_* \left(-A \sin\alpha + B \cos\alpha\right), \quad \tilde{C} = \phi_* (C)\,. \tag{2.12}$$

(ii) *Two hyperbolic systems Ξ_H and $\tilde{\Xi}_H$ are feedback equivalent if and only if there exists a diffeomorphism $\tilde{x} = \phi(x)$ and an invertible reparametrisation (nonlinear feedback) $w = \psi(x, \tilde{w})$, of the form $\psi = \pm\tilde{w} + \alpha(x)$, and satisfying*

$$\tilde{A} = \phi_* \left(A \cosh\alpha + B \sinh\alpha\right), \quad \tilde{B} = \pm\phi_* \left(A \sinh\alpha + B \cosh\alpha\right), \quad \tilde{C} = \phi_* (C)\,. \tag{2.13}$$

(iii) *Two parabolic systems Ξ_P and $\tilde{\Xi}_P$ are feedback equivalent if and only if there exists a diffeomorphism $\tilde{x} = \phi(x)$ and an invertible reparametrisation (nonlinear feedback) $w = \psi(x, \tilde{w})$, of the form $\psi = \alpha(x) + \beta(x)\tilde{w}$ and $\beta(\cdot) \neq 0$, and satisfying*

$$\tilde{A} = \phi_* \left(A\beta^2\right), \quad \tilde{B} = \phi_* \left(2A\alpha\beta + B\beta\right), \quad \tilde{C} = \phi_* \left(C + A\alpha^2 + B\alpha\right). \tag{2.14}$$

Proof. We show the necessity of each statement as the converse implications are immediate. In all three cases, it is clear that diffeomorphisms of $\mathcal{X}$ map quadratic systems into quadratic systems; Hence, we only show that pure feedback transformations $w = \psi(x, \tilde{w})$ are of the stated form.

(i) Assume that Ξ_E and $\tilde{\Xi}_E$ are equivalent via a reparametrisation $w = \psi(x, \tilde{w})$. Then, we have the following relation $f_E(x, \psi(x, \tilde{w})) = \tilde{f}_E(x, \tilde{w})$, which we differentiate three times with respect to $\tilde{w}$ and using $\frac{\partial^3 \tilde{f}_E}{\partial \tilde{w}^3} = -\frac{\partial \tilde{f}_E}{\partial \tilde{w}}$, we conclude the relation $\frac{\partial^3 f_E}{\partial \tilde{w}^3} = -\frac{\partial f_E}{\partial \tilde{w}}$, which translates into

$$\begin{aligned} &A\left(-\psi''' \sin(\psi) + (\psi')^3 \sin(\psi) - 3\psi'\psi'' \cos(\psi)\right) \\ &+ B\left(\psi''' \cos(\psi) - (\psi')^3 \cos(\psi) - 3\psi'\psi'' \sin(\psi)\right) = A\psi' \sin(\psi) - B\psi' \cos(\psi), \end{aligned}$$

where the derivatives are taken with respect to $\tilde{w}$. Since the functions cos and sin are independent, we obtain $\psi'' = 0$ and $(\psi')^2 = 1$. Thus $\psi(x, \tilde{w}) = \pm\tilde{w} + \alpha(x)$. Applying this reparametrisation (together with a diffeomorphism ϕ) to Ξ_E, we obtain the relations of (2.12).

(ii) The first part of the reasoning is exactly the same, using f_H and the fact $\frac{\partial^3 f_H}{\partial \tilde{w}^3} = \frac{\partial f_H}{\partial \tilde{w}}$. Applying the reparametrisation $w = \psi(x, \tilde{w}) = \pm\tilde{w} + \alpha$ (together with a diffeomorphism ϕ) to Ξ_H, we obtain the relations of (2.13).

(iii) We repeat again the same reasoning to f_P with the property $\frac{\partial^3}{\partial \tilde{w}^3} f_P = 0$. However, this time we obtain the conditions $\psi''' = 0$ and $\psi\psi''' + 3\psi'\psi'' = 0$ on the reparametrisation ψ, which implies $\psi'' = 0$, that is $\psi(x, \tilde{w}) = \beta(x)\tilde{w} + \alpha(x)$, with β satisfying $\beta(\cdot) \neq 0$. Applying this reparametrisation (together with a diffeomorphism ϕ) to Ξ_P, yields the relations of (2.14).

$\blacksquare$

Remark. Initially, Ξ_Q was considered locally around a point (x_0, w_0), however, by the last proposition, the transformations $w = \psi(x, \tilde{w})$ are *global* with respect to w, so we will consider the systems Ξ_Q locally in x and globally with respect to w. All results below are stated assuming this structure. $\blacklozenge$

We will develop relations involving structure functions uniquely attached to any fixed triple (A, B, C) and thus change accordingly with diffeomorphisms $\tilde{x} = \phi(x)$. So we will act on (A, B, C) by (α, β) only (β is ± 1 in the elliptic and hyperbolic cases) and we will denote by $(\tilde{A}, \tilde{B}, \tilde{C})$ the result of that action (given by (2.12), or (2.13) or (2.14), with $\phi = \mathrm{id}$), called a reparametrisation.
Observe that the reparametrisations of Ξ_P depend on two smooth functions while those of Ξ_E and Ξ_H depend on one smooth function only. Therefore we expect the classification of parabolic systems to be less rich (less parametrising functions) than the classification of elliptic and hyperbolic systems. For the elliptic (resp. hyperbolic) case, in order to avoid unnecessary computations, we assume that E-frames, resp. H-frames, (A, B) and $(\tilde{A}, \tilde{B})$ of two equivalent systems have the same orientation (we will come back to this simplification in Proposition 2.5). Thus we restrict reparametrisations (2.12), resp. (2.13), to those with $\beta = 1$. In the following subsections we will first classify together elliptic and hyperbolic systems as the procedures are similar, and then we will classify parabolic systems.

3.1 Classification of elliptic and hyperbolic systems

In this subsection, we start classifying elliptic and hyperbolic systems under the action of reparametrisations. Firstly, we will give a prenormal form for both types of systems showing that they actually depend on three smooth functions. Secondly, we will further develop their classification, in particular we will give conditions for the existence of commutative frames and a complete characterisation of forms without functional parameters.

Notations. In order to simplify notations, in the following formulae the upper sign corresponds to the elliptic case and the lower sign to the hyperbolic case, e.g. we will use the symbol $\pm$ to design similar objects attached to the elliptic ($+$ case) and to the hyperbolic ($-$ case) systems and in the case of a $\mp$ symbol we have $-$ for elliptic systems and $+$ for hyperbolic ones. We denote Ξ_{EH} elliptic and hyperbolic systems, and an EH-frame stands for an E-frame or an H-frame. Denoting by $\bar{R}_{EH}(\alpha)$ the (trigonometric or hyperbolic) rotation matrix given by

$$\bar{R}_E(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \quad \text{and} \quad \bar{R}_H(\alpha) = \begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{pmatrix},$$

respectively, we see, from (2.12) and (2.13), that EH-frames are transformed by $(\tilde{A}, \tilde{B}) = (A, B)\bar{R}_{EH}(\pm\alpha)$ under reparametrisations. Introduce structure functions (μ_0, μ_1) and (γ_0, γ_1) uniquely defined by $[A, B] = \mu_0 A + \mu_1 B$ and $C = \gamma_0 A + \gamma_1 B$, respectively. We denote $\gamma = (\gamma_0, \gamma_1)$, and $\rotatebox{90}{$\gamma$} = (\gamma_1, \mp\gamma_0)$, and set $\Gamma_{EH} = (\gamma_0)^2 \pm (\gamma_1)^2$. We begin by a technical lemma showing how structure functions behave under reparametrisations.

Lemma 2.2 (Transformation of structure functions). *Consider an elliptic/hyperbolic system Ξ_{EH} with structure functions $(\mu_0, \mu_1, \gamma_0, \gamma_1)$. Then under the reparametrisation $w = \tilde{w} + \alpha(x)$ we have,*

$$(\tilde{\mu_0}, \tilde{\mu_1}) = (\mu_0 \mp \mathrm{L}_A(\alpha), \mu_1 - \mathrm{L}_B(\alpha))\, \bar{R}_{EH}(\alpha), \quad \textit{and} \quad \tilde{\gamma} = \gamma\, \bar{R}_{EH}(\alpha). \tag{2.15}$$

Proof. Details of the computations can be found in Appendix 2.C. ■

Moreover, it is a straightforward calculation using (2.15) to see that Γ_{EH} is invariant under reparametrisations.

Proposition 2.3 (Prenormal form of elliptic and hyperbolic systems).

(i) Any elliptic system Ξ_E, resp. hyperbolic system Ξ_H, always admits under a feedback transformation the following prenormal form,

$$\Xi_E^{pn} : \dot{x} = r(x)\begin{pmatrix}\cos(w)\\ \sin(w)\end{pmatrix} + \begin{pmatrix}c_0(x)\\ c_1(x)\end{pmatrix}, \ \textit{resp.}\ \Xi_H^{pn} : \dot{x} = r(x)\begin{pmatrix}\cosh(w)\\ \sinh(w)\end{pmatrix} + \begin{pmatrix}c_0(x)\\ c_1(x)\end{pmatrix},$$

with $r > 0$.

(ii) Two prenormal forms Ξ_E^{pn} and $\tilde{\Xi}_E^{pn}$, resp. Ξ_H^{pn} and $\tilde{\Xi}_H^{pn}$, are feedback equivalent if and only if there exists a diffeomorphism $\tilde{x} = \phi(x) = (\phi_1(x), \phi_2(x))$ satisfying

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial y},\ \frac{\partial \phi_1}{\partial y} = \mp\frac{\partial \phi_2}{\partial z},\quad \left(\frac{\partial \phi_1}{\partial z}\right)^2 \pm \left(\frac{\partial \phi_1}{\partial y}\right)^2 = \left(\frac{\tilde{r}}{r}\right)^2, \ \textit{and}\quad \phi_* C = \tilde{C}. \tag{2.16}$$

Notice that the prenormal forms Ξ_E^{pn} and Ξ_H^{pn} are parametrisations of the elliptic and hyperbolic submanifolds $\mathcal{S}_E$ and $\mathcal{S}_H$ given in Lemma 2.1, and thus, the first statement of the above proposition is equivalent to the statements *(i)* and *(ii)* of that lemma.

Proof.

(i) For the system $\Xi_{EH} = (A, B, C)$, define a (pseudo-)Riemannian metric $\mathsf{g}_\pm$ on $\mathcal{X}$ by declaring $\mathsf{g}_\pm(A, A) = 1$, $\mathsf{g}_\pm(B, B) = \pm 1$, and $\mathsf{g}_\pm(A, B) = 0$. It is known that any non-degenerate metric on a manifold of dimension two is conformally flat (see [Ber58, pp 15-35] or [Spi99b, Addendum 1 of chapter 9] for the elliptic case and [Sch08, theorem 7.2] for the hyperbolic case). Therefore, there exists a diffeomorphism $(\tilde{z}, \tilde{y}) = \tilde{x} = \phi(x)$ such that $\mathsf{g}_\pm = \phi^* \tilde{\mathsf{g}}_\pm$, where $\tilde{\mathsf{g}}_\pm = \varrho(\tilde{x})\,(d\tilde{z}^2 \pm d\tilde{y}^2)$, $\varrho > 0$. The vector fields $\tilde{A} = \phi_* A$ and $\tilde{B} = \phi_* B$ satisfy $\tilde{\mathsf{g}}_\pm(\tilde{A}, \tilde{A}) = 1$, $\tilde{\mathsf{g}}_\pm(\tilde{B}, \tilde{B}) = \pm 1$, and $\tilde{\mathsf{g}}_\pm(\tilde{A}, \tilde{B}) = 0$ which implies that $(\tilde{A}, \tilde{B})$ is a (pseudo-)orthonormal frame. Finally, using feedback α we can smoothly *rotate* $(\tilde{A}, \tilde{B})$ into $\left(r\frac{\partial}{\partial \tilde{z}}, r\frac{\partial}{\partial \tilde{y}}\right)$ with $r = \frac{1}{\sqrt{\varrho}}$, which gives the desired form $\tilde{\Xi}_{EH} = (\tilde{A}, \tilde{B}, \tilde{C})$, with $\tilde{C} = \phi_* C$.

(ii) By relations (2.12) and (2.13), the reparametrisations do not act on C and thus the relation $\tilde{C} = \phi_* C$ is necessary for the equivalence of prenormal forms. Consider two elliptic prenormal forms Ξ_E^{pn} and $\tilde{\Xi}_E^{pn}$, resp. two hyperbolic prenormal

forms Ξ^{pn}_H and $\tilde{\Xi}^{pn}_H$, related by a feedback $w = \tilde{w} + \alpha$ and a diffeomorphism ϕ. Thus, by adapting relation (2.12), resp. (2.13), we obtain

$$\frac{\partial \phi_1}{\partial z} = \frac{\tilde{r}}{r}\cos(\alpha) = \frac{\partial \phi_2}{\partial y}, \quad \frac{\partial \phi_1}{\partial y} = \frac{\tilde{r}}{r}\sin(\alpha) = -\frac{\partial \phi_2}{\partial z},$$
$$\text{resp.} \quad \frac{\partial \phi_1}{\partial z} = \frac{\tilde{r}}{r}\cosh(\alpha) = \frac{\partial \phi_2}{\partial y}, \quad \frac{\partial \phi_1}{\partial y} = -\frac{\tilde{r}}{r}\sinh(\alpha) = \frac{\partial \phi_2}{\partial z},$$

from which we deduce condition (2.16). Conversely, applying the diffeomorphism ϕ given by (2.16), together with the feedback $w = \tilde{w} + \alpha$, with α being a solution of

$$\cos(\alpha) = \frac{r}{\tilde{r}}\frac{\partial \phi_1}{\partial z}, \quad \sin(\alpha) = -\frac{r}{\tilde{r}}\frac{\partial \phi_1}{\partial y},$$
$$\text{resp.} \quad \cosh(\alpha) = \frac{r}{\tilde{r}}\frac{\partial \phi_1}{\partial z}, \quad \sinh(\alpha) = -\frac{r}{\tilde{r}}\frac{\partial \phi_1}{\partial y},$$

we transform Ξ^{pn}_E into $\tilde{\Xi}^{pn}_E$, resp. Ξ^{pn}_H into $\tilde{\Xi}^{pn}_H$.

■

Remark. In the above proof we used the metric $\mathsf{g}_\pm$ on $\mathcal{X}$ defined by

$$\mathsf{g}_\pm(A,A) = 1, \quad \mathsf{g}_\pm(B,B) = \pm 1, \quad \mathsf{g}_\pm(A,B) = 0. \tag{2.17}$$

This object will play a special role in the interpretation of the conditions describing the existence of a commutative EH-frame. ◆

The above proposition shows that elliptic and hyperbolic systems are parametrised by three smooth functions (and not by 6 defining the triple (A, B, C)) of two variables. Moreover, relation (2.16) shows that the group of diffeomorphisms, conjugating the EH-frames of two given prenormal forms is parametrised by any function ϕ_1 of two variables satisfying the third equation of (2.16) (ϕ_2 being given in terms of ϕ_1 via the first and second equations) The following proposition gives equivalent algebraic and geometric conditions for the existence of a commutative EH-frame.

Proposition 2.4 (Existence of a commutative EH-frame)**.** *Consider an elliptic/hyperbolic system $\Xi_{EH} = (A, B, C)$ with structure functions (μ_0, μ_1) of the EH-frame (A, B). The following statements are equivalent*

(i) There exists a commutative EH-frame.

(ii) The structure functions (μ_0, μ_1) attached to the EH-frame (A, B) satisfy

$$-(\mu_0)^2 \mp (\mu_1)^2 \pm \mathrm{L}_A(\mu_1) - \mathrm{L}_B(\mu_0) = 0. \tag{2.18}$$

(iii) The Gaussian curvature $\kappa_\pm$ of the metric $\mathsf{g}_\pm$ vanishes.

Notice that statement *(i)* describes the following normal forms,

$$\Xi'_E : \begin{cases} \dot{z} &= \cos(w) + c_0(x) \\ \dot{y} &= \sin(w) + c_1(x) \end{cases}, \quad \text{and} \quad \Xi'_H : \begin{cases} \dot{z} &= \cosh(w) + c_0(x) \\ \dot{y} &= \sinh(w) + c_1(x) \end{cases},$$

whose structure functions are $\mu_0 = \mu_1 = 0$, $\gamma_0 = c_0$, and $\gamma_1 = c_1$. We call Ξ'_E a *flat elliptic system* and Ξ'_H a *flat hyperbolic system.*

Proof. The equivalence between *(ii)* and *(iii)* is immediate since (2.18) is the Gaussian curvature $\kappa_{\pm}$ of $\mathsf{g}_{\pm}$ (details of the computations are in Appendix 2.D). We show that *(i)* is equivalent to *(ii)*. If the EH-frame (A, B) is equivalent via $w = \tilde{w} + \alpha(x)$ to a commutative EH-frame $(\tilde{A}, \tilde{B})$, then by (2.15) we immediately have $\mathrm{L}_A(\alpha) = \pm\mu_0$ and $\mathrm{L}_B(\alpha) = \mu_1$; the integrability condition of this system of first order partial differential equations gives (2.18). Conversely, construct α as a solution of the system $\mathrm{L}_A(\alpha) = \pm\mu_0$ and $\mathrm{L}_B(\alpha) = \mu_1$, whose solvability is guaranteed by the integrability condition given by (2.18). Then by (2.15) we get that the resulting EH-frame $(\tilde{A}, \tilde{B})$ is commutative. ■

Notice that when proving Proposition 2.4 we have shown that the Gaussian curvature $\kappa_{\pm}$ of the metric $\mathsf{g}_{\pm}$ is given by the left hand side of (2.18). Moreover relation (2.15) implies that $\kappa_{\pm}$ is invariant under reparametrisations $w = \tilde{w} + \alpha$ and is therefore an equivariant of the feedback transformations of the system Ξ_{EH}.

Remark. Since we have $\kappa_{\pm} = \phi^* \tilde{\kappa}_{\pm}$ under feedback transformations, an interesting generalisation of the above proposition is a description of elliptic/hyperbolic systems having a constant Gaussian curvature $\kappa_{\pm}$. A simple analysis can be performed using the prenormal forms Ξ_{EH}^{pn} and is presented in the sequel. Consider a prenormal form Ξ_{EH}^{pn} for which we have the EH-frame $(A, B) = \left(r\frac{\partial}{\partial z}, r\frac{\partial}{\partial y}\right)$ and the associated structure functions $\mu_0 = -\frac{\partial r}{\partial y}$ and $\mu_1 = \frac{\partial r}{\partial z}$. The Gaussian curvature $\kappa_{\pm}$ of the metric $\mathsf{g}_{\pm} = \frac{1}{r^2}\left(dz^2 \pm dy^2\right)$ is given by

$$\pm\Delta_{\pm}(\ln(r)) = \kappa_{\pm}, \tag{2.19}$$

where $\Delta_{\pm} = r^2\left(\frac{\partial^2}{\partial z^2} \pm \frac{\partial^2}{\partial y^2}\right)$ is the Laplace-Beltrami operator associated to that metric. If $\kappa_{\pm}$ is constant, then (2.19) is called the Liouville equation and, in the elliptic and hyperbolic cases, it admits solutions described as follows. In the elliptic case, introduce the new variables $(\mathbf{x}, \bar{\mathbf{x}}) = (z + \mathbf{i}y, z - \mathbf{i}y)$ and for the new unknown function $\lambda = \frac{1}{r^2}$ we obtain the equation $\frac{\partial^2}{\partial \mathbf{x} \partial \bar{\mathbf{x}}} \ln(\lambda) = -2\kappa_+ \lambda$. Following [Hen93] and [PP17], solutions of that equation are $\lambda = \frac{r_1'(\mathbf{x})\overline{r_1}'(\bar{\mathbf{x}})}{\left(1+\kappa_+ r_1(\mathbf{x})\overline{r_1(\mathbf{x})}\right)^2}$, where $r_1(\mathbf{x})$ is any holomorphic function satisfying $r_1'(\mathbf{x}_0) \neq 0$. These solutions yield all prenormal forms Ξ_E^{pn} with constant curvature κ_+ to be given by $r(x) = r_E(z, y)$, where

$$r_E(z, y)^2 = \frac{\left(1 + \kappa_+ r_1(z + \mathbf{i}y)\overline{r_1}(z - \mathbf{i}y)\right)^2}{r_1'(z + \mathbf{i}y)\overline{r_1}'(z - \mathbf{i}y)}.$$

In the hyperbolic case, introduce coordinates $(\tilde{z}, \tilde{y}) = (z + y, z - y)$ and the new unknown function $\lambda = \frac{1}{r^2}$ so that (2.19) becomes now $\frac{\partial^2}{\partial \tilde{z} \partial \tilde{y}} \ln(\lambda) = 2\kappa_- \lambda$. Adapting [Lio53, formula (3)], we obtain the general solution $\lambda = \frac{r_1'(\tilde{z})r_2'(\tilde{y})}{(1 - \kappa_- r_1(\tilde{z})r_2(\tilde{y}))^2}$, where r_1 and r_2 are any smooth functions satisfying $r_1'(\tilde{z}_0)r_2'(\tilde{y}_0) > 0$. These solutions give $r(x) = r_H(z, y)$ for all prenormal forms Ξ_H^{pn} with constant curvature κ_-, where

$$r_H(z, y)^2 = \frac{\left(1 - \kappa_- r_1(z + y)r_2(z - y)\right)^2}{r_1'(z + y)r_2'(z - y)}.$$

◆

In the following proposition, we give first a classification of flat elliptic/hyperbolic systems, second we characterise those without functional parameters, and third we provide a canonical form for the latter class. Recall that $\wr = (\gamma_1, \mp\gamma_0)$ and that for flat elliptic/hyperbolic systems Ξ'_{EH} we have $(\gamma_0, \gamma_1) = (c_0, c_1)$ so all statements of the proposition below are actually expressed in terms of structure functions. From now on, we will consider the full group of feedback transformations consisting of $\tilde{x} = \phi(x)$ and $w = \pm\tilde{w} + \alpha(x)$. The additional transformation $w = -\tilde{w} + \alpha$ implies $(\tilde{A}, \tilde{B}) = (A, B)\bar{\bar{R}}_{EH}(\pm\alpha)$ and the corresponding structure functions change (compare (2.15)) by

$$(2.15') \qquad (\tilde{\mu}_0, \tilde{\mu}_1) = -(\mu_0 \mp \mathrm{L}_A(\alpha), \mu_1 - \mathrm{L}_B(\alpha))\bar{\bar{R}}_{EH}(\alpha) \quad \text{and} \quad \tilde{\gamma} = \gamma\bar{\bar{R}}_{EH}(\alpha),$$

where $\bar{\bar{R}}_E(\alpha) = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix}$ and $\bar{\bar{R}}_H(\alpha) = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ -\sinh(\alpha) & -\cosh(\alpha) \end{pmatrix}$.

Proposition 2.5 (Classification of flat elliptic/hyperbolic systems)**.**

(i) *Two flat elliptic systems Ξ'_E and $\tilde{\Xi}'_E$, resp. two flat hyperbolic systems Ξ'_H and $\tilde{\Xi}'_H$, are locally equivalent around $x_0 = 0 \in \mathbb{R}^2$ if and only if there exists a constant $\alpha \in \mathbb{R}$ satisfying*

$$(2.20) \qquad R_{EH}(\pm\alpha)^{-1}C(x) = \tilde{C}\left(R_{EH}(\pm\alpha)^{-1}x\right),$$

where R_{EH} stands for either $\bar{R}_{EH}$ or $\bar{\bar{R}}_{EH}$.

(ii) *An elliptic/hyperbolic system Ξ_{EH} is locally equivalent to Ξ'_{EH} with $(c_0, c_1) \in \mathbb{R}^2$ if and only if one of the equivalent conditions of Proposition 2.4 holds and, additionally,*

$$(2.21) \qquad \mathrm{L}_A(\gamma) + \wr\mu_0 = 0, \quad \mathrm{L}_B(\gamma) \pm \wr\mu_1 = 0.$$

(iii) *A flat elliptic system Ξ'_E with $(c_0, c_1) \in \mathbb{R}^2$ is always feedback equivalent, locally around $x_0 = 0 \in \mathbb{R}^2$, to the canonical form*

$$\Xi_E^{\Gamma_E} : \begin{cases} \dot{z} &= \cos(w) + \sqrt{\Gamma_E} \\ \dot{y} &= \sin(w) \end{cases},$$

where $\Gamma_E = (c_0)^2 + (c_1)^2$ is an invariant.

(iv) *A flat hyperbolic system Ξ'_H with $(c_0, c_1) \in \mathbb{R}^2$ is always feedback equivalent, locally around $x_0 = 0 \in \mathbb{R}^2$, to one of the following canonical form*

$$\Xi_H^{\Gamma_H,\varepsilon} : \begin{cases} \dot{z} &= \cosh(w) + \varepsilon\sqrt{\Gamma_H} \\ \dot{y} &= \sinh(w) \end{cases}, \quad \text{or} \quad \Xi_H^{-\Gamma_H} : \begin{cases} \dot{z} &= \cosh(w) \\ \dot{y} &= \sinh(w) + \sqrt{-\Gamma_H} \end{cases},$$

$$\text{or} \quad \Xi_H^{0,\varepsilon} : \begin{cases} \dot{z} &= \cosh(w) + \varepsilon \\ \dot{y} &= \sinh(w) + 1 \end{cases}, \quad \text{or} \quad \Xi_H^{0,0} : \begin{cases} \dot{z} &= \cosh(w) \\ \dot{y} &= \sinh(w) \end{cases},$$

where $\Gamma_H = (c_0)^2 - (c_1)^2$ and satisfies $\Gamma_H > 0$ for the first form, $\Gamma_H < 0$ for the second form, and $\Gamma_H = 0$ for the third and fourth ones, where $\varepsilon = \operatorname{sgn}(c_0) = \pm 1$. Moreover (Γ_H, ε) is a complete invariant.

Remark. In statement *(iv)*, notice that there are two orbits of the local action of feedback transformations group for $\Gamma_H > 0$, corresponding to $\operatorname{sgn}(c_0) = \varepsilon = \pm 1$, one orbit for $\Gamma_H < 0$, and three orbits for $\Gamma_H = 0$ corresponding, respectively, to $\operatorname{sgn}(c_0) = \varepsilon = \pm 1$ or $(c_0, c_1) = (0, 0)$. The invariant $\varepsilon = \pm 1$ corresponds to the parametrisation of one of two branches of the hyperbola $(\dot{z} - \sqrt{\Gamma_H})^2 - \dot{y}^2 = 1$. ◆

Proof.

(i) Consider, locally around $0 \in \mathbb{R}^2$, two equivalent flat elliptic/hyperbolic systems Ξ'_{EH} and $\tilde{\Xi}'_{EH}$ given by structure functions $(\mu_0, \mu_1, \gamma_0, \gamma_1) = (0, 0, c_0, c_1)$ and $(\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\gamma}_0, \tilde{\gamma}_1) = (0, 0, \tilde{c}_0, \tilde{c}_1)$, respectively. Since they both have a commutative EH-frame, by (2.15) (and (2.15')) they differ by a reparametrisation $w = \pm\tilde{w} + \alpha$ satisfying $\mathrm{L}_A(\alpha) = \mathrm{L}_B(\alpha) = 0$ and thus $\alpha \in \mathbb{R}$. Applying this reparametrisation together with a diffeomorphism ϕ satisfying $\phi_* = R_{EH}(\pm\alpha)^{-1}$, that is $\tilde{x} = \phi(x) = R_{EH}(\pm\alpha)^{-1}x$, transforms Ξ'_{EH} into $\tilde{\Xi}'_{EH}$ if and only if

$$\begin{pmatrix} \tilde{c}_0(\tilde{x}) \\ \tilde{c}_1(\tilde{x}) \end{pmatrix} = R_{EH}(\pm\alpha)^{-1} \begin{pmatrix} c_0(x) \\ c_1(x) \end{pmatrix}.$$

(ii) Assume that Ξ_{EH}, given by structure functions $(\mu_0, \mu_1, \gamma_0, \gamma_1)$, is equivalent via $\tilde{x} = \phi(x)$ and $w = \tilde{w} + \alpha$ to Ξ'_{EH} with structure functions $(\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\gamma}_0, \tilde{\gamma}_1) = (0, 0, c_0, c_1)$, where $(c_0, c_1) \in \mathbb{R}^2$. Necessity of one (and thus any) of the conditions of Proposition 2.4 is clear, and by (2.15) and (2.15') we have first, $\mathrm{L}_A(\alpha) = \pm\mu_0$ and $\mathrm{L}_B(\alpha) = \mu_1$ and second, $\gamma R_{EH}(\alpha) = \tilde{\gamma} = (c_0, c_1)$. By differentiating this last relation along A and B we obtain

$$\begin{aligned} &\mathrm{L}_A(\gamma)\, R_{EH}(\alpha) + \gamma \mathrm{L}_A(R_{EH}(\alpha)) = \\ &\qquad \mathrm{L}_A(\gamma)\, R_{EH}(\alpha) + \gamma \left(\pm\mathrm{L}_A(\alpha) \begin{pmatrix} 0 & \mp 1 \\ 1 & 0 \end{pmatrix} R_{EH}(\alpha) \right) = \mathrm{L}_A(\gamma) + \llcorner\mu_0 = 0, \end{aligned}$$

and

$$\begin{aligned} &\mathrm{L}_B(\gamma)\; R_{EH}(\alpha) + \gamma \mathrm{L}_B(R_{EH}(\alpha)) = \\ &\qquad \mathrm{L}_B(\gamma)\, R_{EH}(\alpha) + \gamma \left(\pm\mathrm{L}_B(\alpha) \begin{pmatrix} 0 & \mp 1 \\ 1 & 0 \end{pmatrix} R_{EH}(\alpha) \right) = \mathrm{L}_B(\gamma) \pm \llcorner\mu_1 = 0, \end{aligned}$$

thus proving that condition (2.21) is necessary. Conversely, assume that (2.18) and (2.21) hold for Ξ_{EH}. By Proposition 2.4, Ξ_{EH} is equivalent to Ξ'_{EH} with a commutative EH-frame, and applying (2.21) to the latter we get $\mathrm{L}_A(\gamma) = \mathrm{L}_B(\gamma) = 0$ and, therefore, we have $(c_0, c_1) \in \mathbb{R}^2$.

(iii) Consider a flat elliptic system Ξ'_E with $(c_0, c_1) \in \mathbb{R}^2$, then relation (2.20) reads

$$\text{(2.20')} \qquad \begin{pmatrix} \tilde{c}_0 \\ \tilde{c}_1 \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \mp\sin(\alpha) & \pm\cos(\alpha) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}.$$

Take α as a solution of $-\sin(\alpha)c_0 + \cos(\alpha)c_1 = 0$, then we have $\tilde{c}_1 = 0$ and $\tilde{c}_0 = \pm\sqrt{\Gamma_E}$, with $\Gamma_E = (c_0)^2 + (c_1)^2$. If necessary, apply $\alpha = \pi$ to send $(\tilde{c}_0, \tilde{c}_1) = (-\sqrt{\Gamma_E}, 0)$ into $(\sqrt{\Gamma_E}, 0)$. The proof that $\Xi_E^{\Gamma_E}$ is equivalent to $\tilde{\Xi}_E^{\tilde{\Gamma}_E}$ if and only if $\Gamma_E = \tilde{\Gamma}_E$ is immediate from (2.20').

(iv) Consider a flat hyperbolic system Ξ'_H with $(c_0, c_1) \in \mathbb{R}^2$ and denote $\Gamma_H = (c_0)^2 - (c_1)^2$, then relation (2.20) reads

$$\begin{pmatrix} \tilde{c}_0 \\ \tilde{c}_1 \end{pmatrix} = \begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ \mp\sinh(\alpha) & \pm\cosh(\alpha) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}. \tag{2.20''}$$

We consider four cases. First, assume that $\Gamma_H > 0$, that is $c_0 \neq 0$ and $-1 < \frac{c_1}{c_0} < 1$, and take α as a solution of $\tanh(\alpha) = \frac{c_1}{c_0}$. Applying the reparametrisation $w = \tilde{w} + \alpha$ yields $\tilde{c}_1 = 0$ and $\tilde{c}_0 = \operatorname{sgn}(c_0)\sqrt{\Gamma_H}$, namely, canonical form $\Xi_H^{\Gamma_H,\varepsilon}$. Second, assume that $\Gamma_H < 0$, that is $c_1 \neq 0$ and $-1 < \frac{c_0}{c_1} < 1$, and take α as a solution of $\tanh(\alpha) = \frac{c_0}{c_1}$, which leads to $\tilde{c}_0 = 0$ and $\tilde{c}_1 = \operatorname{sgn}(c_1)\sqrt{-\Gamma_H}$. If $\operatorname{sgn}(c_1) = -1$, then by applying (2.20'') with $\alpha = 0$ and the bottom sign, we can always normalise $\operatorname{sgn}(c_1)$ yielding canonical form $\Xi_H^{-\Gamma_H}$. Third, assume that $\Gamma_H = 0$ and $c_0 = 0$ thus $c_1 = 0$ and therefore we immediately have the canonical form $\Xi_H^{0,0}$. Fourth, and finally, assume that $\Gamma_H = 0$ and $c_0 \neq 0$, thus $c_1 = \varepsilon c_0$ with $\varepsilon = \pm 1$. If necessary, apply (2.20'') with $\alpha = 0$ and the bottom sign to obtain $c_1 > 0$. Take $\alpha = \varepsilon \ln c_1$ and apply (2.20'') with the upper sign to obtain $\tilde{c}_1 = 1$ and $\tilde{c}_0 = \varepsilon$. To show that (Γ_H, ε) is a complete invariant is trivial by applying (2.20'') to the canonical forms $\Xi_H^{\Gamma_H,\varepsilon}$, $\Xi_H^{-\Gamma_H}$, $\Xi_H^{0,\varepsilon}$, and $\Xi_H^{0,0}$.

■

The following corollary gives the corresponding normal and canonical forms for elliptic and hyperbolic submanifolds. To this end, consider a conic submanifold $\mathcal{S}_q$ given by the triple (g, ω, h), where g is a (pseudo-)Riemanian metric, ω is a differential one form, and h is a function, and assume that $\mathcal{S}_q$ satisfies $\Delta_1 \neq 0$ and $\Delta_2 \neq 0$ (see the paragraph before Lemma 2.1 for the definition of Δ_1 and Δ_2). The triple $\mathcal{S}_q$ is given up to the conformal equivalence $(\delta\mathsf{g}, \delta\omega, \delta h)$, with $\delta(x) \neq 0$. We show how to construct (in a canonical way) a normalised triple on which we can test the conditions of the corollary below. To this aim, define the vector field $v = \omega^\sharp$, where $\sharp$ is the canonical musical isomorphism associated to the metric g (see e.g. [Lee13, p. 192-193], for a definition). We claim that the function $\varrho := \mathsf{g}(v, v) - h$ satisfies $\varrho(x) \neq 0$. Indeed, for $\mathsf{g} = (\mathsf{g}_{ij})$ and $\omega = (\omega_1, \omega_2)$ we have

$$\Delta_1 = \Delta_2 h + 2\mathsf{g}_{12}\omega_1\omega_2 - \mathsf{g}_{11}(\omega_2)^2 - \mathsf{g}_{22}(\omega_1)^2 = \Delta_2 h - \Delta_2 \mathsf{g}(v, v) \neq 0.$$

Finally, the normalised triple is given by $\mathcal{S}_{EH} = (\mathsf{g}_{EH}, \omega_{EH}, h_{EH}) = \left(\frac{1}{\varrho}\mathsf{g}, \frac{1}{\varrho}\omega, \frac{1}{\varrho}h\right)$ and is characterised by the normalising condition $\mathsf{g}_{EH}(v, v) - h_{EH} \equiv 1$. Notice that v is not affected by the conformal transformations.

In the following, we denote $\Gamma_{EH} := \mathsf{g}_{EH}(v, v)$ (see a justification below Corollary 2.3), we call $\mathcal{S}_q$ a *flat* submanifold if the metric g_{EH} is flat, and we call $\mathcal{S}_q$ a *strongly flat* submanifold if in well chosen coordinates the polynomial S_q, of degree 2 with respect to $\dot{x}$, has constant coefficients (so, in particular, g_{EH} is flat).

Corollary 2.3 (Normal and canonical forms of elliptic/hyperbolic submanifolds). *For an elliptic/hyperbolic submanifold given in terms of the normalised triple $\mathcal{S}_{EH} = (\mathsf{g}_{EH}, \omega_{EH}, h_{EH})$, the following statements hold:*

(i) $\mathcal{S}_q$ is flat if and only if the Gaussian curvature of g_{EH} vanishes.

(ii) $\mathcal{S}_q$ *is strongly flat if and only if the Gaussian curvature of* $\mathtt{g}_{EH}$ *vanishes and, additionally,*

$$\mathcal{L}_A\omega_{EH} = \mathcal{L}_B\omega_{EH} = 0, \tag{2.22}$$

where $\mathcal{L}$ *denotes the Lie derivative of differential forms, and* (A, B) *is any (pseudo-)orthonormal frame associated to* $\mathtt{g}_{EH}$*, i.e.* $\mathtt{g}_{EH}(A, A) = 1$*,* $\mathtt{g}_{EH}(B, B) = \pm 1$*, and* $\mathtt{g}_{EH}(A, B) = 0$*.*

(iii) *An elliptic strongly flat submanifold* $\mathcal{S}_q$ *always admits the following canonical form*

$$\mathcal{S}_E^{\Gamma_E} = \{(\dot{z} - \sqrt{\Gamma_E})^2 + \dot{y}^2 = 1\} = \left(dz^2 + dy^2, -\sqrt{\Gamma_E}dz, \Gamma_E - 1\right)$$

(iv) *A hyperbolic strongly flat submanifold* $\mathcal{S}_q$ *always admits one of the following canonical forms*

$$\begin{aligned}
\mathcal{S}_H^{\Gamma_H} &= \left\{(\dot{z} - \sqrt{\Gamma_H})^2 - (\dot{y})^2 = 1\right\} &&= \left(dz^2 - dy^2, -\sqrt{\Gamma_H}dz, \Gamma_H - 1\right),\\
\mathcal{S}_H^{-\Gamma_H} &= \left\{(\dot{z})^2 - (\dot{y} - \sqrt{-\Gamma_H})^2 = 1\right\} &&= \left(dz^2 - dy^2, \sqrt{-\Gamma_H}dy, -\Gamma_H - 1\right),\\
\mathcal{S}_H^{0,0} &= \{(\dot{z})^2 - (\dot{y})^2 = 1\} &&= (dz^2 - dy^2, 0, -1),\\
\mathcal{S}_H^{0,1} &= \{(\dot{z} - 1)^2 - (\dot{y} - 1)^2 = 1\} &&= (dz^2 - dy^2, -dz + dy, -1),
\end{aligned}$$

Notice that flat submanifolds correspond to flat elliptic/hyperbolic systems Ξ'_{EH} and that strongly flat submanifolds correspond to the forms of Proposition 2.5. When $\mathcal{S}_{EH}$ is strongly flat, then the invariant Γ_{EH} has the same value as in statements *(iii)* and *(iv)* of Proposition 2.5, which justifies the same notation. Precisely, the canonical form $\mathcal{S}_E^{\Gamma_E}$ describes, in each fiber $T_x\mathcal{X}$, a circle of constant radius 1 translated along the $\frac{\partial}{\partial z}$-direction, and each canonical form $\mathcal{S}_E^{\Gamma_E}$ corresponds to the canonical form $\Xi_E^{\Gamma_E}$. Observe also that the canonical forms $\mathcal{S}_H$ correspond to the canonical form of Ξ_H, namely, $\mathcal{S}_H^{\Gamma_H}$ corresponds to $\Xi_H^{\Gamma_H,\varepsilon}$, $\mathcal{S}_H^{-\Gamma_H}$ corresponds to $\Xi_H^{-\Gamma_H,}$, $\mathcal{S}_H^{0,0}$ corresponds to $\Xi_H^{0,0}$, and $\mathcal{S}_H^{0,1}$ corresponds to $\Xi_H^{0,\varepsilon}$. In the canonical forms of $\mathcal{S}_H$, the invariant ε does not show up (in the case $\Gamma_H > 0$ or $\Gamma_H = 0$ and $c_0 \neq 0$) because the same equation of $\mathcal{S}_H$ describes both branches of the hyperbola.

Proof.

(i) It is a classical result in (pseudo-)Riemannian geometry, see e.g. [Spi99a, Theorem 13, p. 193] or [Lee06, Theorem 7.3] for the elliptic case and [KN63, Theorem 9.1] for the hyperbolic one.

(ii) If $\mathcal{S}_q$ is strongly flat, then by definition in some coordinate system we have $\mathtt{g} = (\mathtt{g}_{ij})$ and $\omega = (\omega_1, \omega_2)$, with $\mathtt{g}_{ij} \in \mathbb{R}$ and $\omega_i \in \mathbb{R}$. Thus in those coordinates a (pseudo-)orthonormal frame (A, B) has also constant coefficients. Therefore, in that coordinate system, $\mathtt{g}$ is flat and condition (2.22) holds but those two conditions do not depend on the choice of coordinates and thus hold in any coordinate system. Conversely we show that $\mathcal{S}_q$, satisfying $\mathtt{g}_{EH}$ is flat and (2.22), is strongly flat. The metric $\mathtt{g}_{EH}$ is flat, therefore in a well-chosen coordinate system, we have $\mathtt{g}_{EH} = dz^2 \pm dy^2$. Condition (2.22) implies that $\omega = \omega_1 dz + \omega_2 dy$ with $\omega_i \in \mathbb{R}$. Finally, by the definition of v, we immediately conclude that v has also constant coefficients and therefore, since $h_{EH} = \mathtt{g}_{EH}(v, v) - 1$, we have $h_{EH} \in \mathbb{R}$.

(iii) In a suitable coordinate system (z, y), a strongly flat elliptic submanifold can be written $\mathcal{S}'_E = \{(\dot{z}-c_0)^2+(\dot{y}-c_1)^2 = 1\}$ with $(c_0, c_1) \in \mathbb{R}^2$ for which we have $\Gamma_E = (c_0)^2 + (c_1)^2$. If $\Gamma_E = 0$, then $(c_0, c_1) = (0, 0)$ and we have the desired form $\mathcal{S}_E^{\Gamma_E=0}$. If $(c_0, c_1) \neq (0, 0)$, then introduce coordinates $(z, y) = \phi(\tilde{z}, \tilde{y})$ whose tangent map is $\phi_* = \begin{pmatrix} \frac{c_0}{\sqrt{\Gamma_E}} & -\frac{c_1}{\sqrt{\Gamma_E}} \\ \frac{c_1}{\sqrt{\Gamma_E}} & \frac{c_0}{\sqrt{\Gamma_E}} \end{pmatrix}$ and we have

$$\begin{aligned}
\left(\frac{c_0}{\sqrt{\Gamma_E}}\dot{\tilde{z}} - \frac{c_1}{\sqrt{\Gamma_E}}\dot{\tilde{y}} - c_0\right)^2 + \left(\frac{c_1}{\sqrt{\Gamma_E}}\dot{\tilde{z}} + \frac{c_0}{\sqrt{\Gamma_E}}\dot{\tilde{y}} - c_1\right)^2 &= 1, \\
(\dot{\tilde{z}})^2 + (\dot{\tilde{y}})^2 + \Gamma_E - 2\frac{(c_0)^2\dot{\tilde{z}}}{\sqrt{\Gamma_E}} - 2\frac{(c_1)^2\dot{\tilde{z}}}{\sqrt{\Gamma_E}} + 2\frac{(c_0)(c_1)\dot{\tilde{y}}}{\sqrt{\Gamma_E}} - 2\frac{(c_0)(c_1)\dot{\tilde{y}}}{\sqrt{\Gamma_E}} &= 1, \\
(\dot{\tilde{z}})^2 + (\dot{\tilde{y}})^2 + \Gamma_E - 2\sqrt{\Gamma_E}\dot{\tilde{z}} = 1, \quad \text{implying} \quad (\dot{\tilde{z}} - \sqrt{\Gamma_E})^2 + (\dot{\tilde{y}})^2 &= 1.
\end{aligned}$$

(iv) In a suitable coordinate system (z, y), a strongly flat hyperbolic submanifold can be written as $\mathcal{S}'_H = \{(\dot{z} - c_0)^2 - (\dot{y} - c_1)^2 = 1\}$ with $(c_0, c_1) \in \mathbb{R}^2$ and for which we have $\Gamma_H = (c_0)^2 - (c_1)^2$. If $\Gamma_H > 0$, introduce coordinates $(z, y) = \phi(\tilde{z}, \tilde{y})$ whose tangent map is $\phi_* = \begin{pmatrix} \frac{c_0}{\sqrt{\Gamma_H}} & \frac{c_1}{\sqrt{\Gamma_H}} \\ \frac{c_1}{\sqrt{\Gamma_H}} & \frac{c_0}{\sqrt{\Gamma_H}} \end{pmatrix}$, and we have

$$\begin{aligned}
\left(\frac{c_0}{\sqrt{\Gamma_H}}\dot{\tilde{z}} + \frac{c_1}{\sqrt{\Gamma_H}}\dot{\tilde{y}} - c_0\right)^2 - \left(\frac{c_1}{\sqrt{\Gamma_H}}\dot{\tilde{z}} + \frac{c_0}{\sqrt{\Gamma_H}}\dot{\tilde{y}} - c_1\right)^2 &= 1, \\
(\dot{\tilde{z}})^2 - (\dot{\tilde{y}})^2 - 2\sqrt{\Gamma_H}\dot{\tilde{z}} + \Gamma_H = 1, \quad \text{implying} \quad (\dot{\tilde{z}} - \sqrt{\Gamma_H})^2 - (\dot{\tilde{y}})^2 &= 1.
\end{aligned}$$

If $\Gamma_H < 0$, introduce coordinates $(z, y) = \phi(\tilde{z}, \tilde{y})$ whose tangent map is $\phi_* = \begin{pmatrix} \frac{c_1}{\sqrt{-\Gamma_H}} & \frac{c_0}{\sqrt{-\Gamma_H}} \\ \frac{c_0}{\sqrt{-\Gamma_H}} & \frac{c_1}{\sqrt{-\Gamma_H}} \end{pmatrix}$, and we have

$$\begin{aligned}
\left(\frac{c_1}{\sqrt{-\Gamma_H}}\dot{\tilde{z}} + \frac{c_0}{\sqrt{-\Gamma_H}}\dot{\tilde{y}} - c_0\right)^2 - \left(\frac{c_0}{\sqrt{-\Gamma_H}}\dot{\tilde{z}} + \frac{c_1}{\sqrt{-\Gamma_H}}\dot{\tilde{y}} - c_1\right)^2 &= 1, \\
(\dot{\tilde{z}})^2 - (\dot{\tilde{y}})^2 + 2\sqrt{-\Gamma_H}\dot{\tilde{y}} + \Gamma_H = 1, \quad \text{implying} \quad (\dot{\tilde{z}})^2 - (\dot{\tilde{y}} - \sqrt{\Gamma_H})^2 &= 1.
\end{aligned}$$

Finally, if $\Gamma_H = 0$, we distinguish two cases. If $c_0 = 0$, then $c_1 = 0$ and there is nothing to do to obtain $\mathcal{S}_H^{0,0}$, otherwise, denote $\varepsilon = \pm 1$ such that $c_0 = \varepsilon c_1$, and take the diffeomorphism $(z, y) = \phi(\tilde{z}, \tilde{y})$ whose tangent map is $\phi_* = \frac{1}{2c_1}\begin{pmatrix} \varepsilon(1 + (c_1)^2) & -\varepsilon(1 + (c_1)^2) \\ (-1 + (c_1)^2) & 1 + (c_1)^2 \end{pmatrix}$ and we have

$$\begin{aligned}
\left(\frac{\varepsilon(1 + (c_1)^2)}{2c_1}\dot{\tilde{z}} + \frac{\varepsilon(-1 + (c_1)^2)}{2c_1}\dot{\tilde{y}} - \varepsilon c_1\right)^2 - \left(\frac{-1 + (c_1)^2}{2c_1}\dot{\tilde{z}} + \frac{1 + (c_1)^2}{2c_1}\dot{\tilde{y}} - c_1\right)^2 &= 1, \\
(\dot{\tilde{z}})^2 - (\dot{\tilde{y}})^2 - 2\tilde{z} + 2\dot{\tilde{y}} = 1, \quad \text{implying} \quad (\dot{\tilde{z}} - 1)^2 - (\dot{\tilde{y}} - 1)^2 &= 1.
\end{aligned}$$

■

To summarise the results of this subsection, we started from a general elliptic/hyperbolic system Ξ_{EH}, which depends on three smooth functions, then with the

Gaussian curvature, associated to the EH-frame (A, B), being zero we reduced the systems to the flat elliptic/hyperbolic systems Ξ'_{EH} depending on two smooth functions only. And finally, we gave conditions characterising the flat systems without functional parameters. In the elliptic case, equivalent systems correspond to the circles $\Gamma_E = (c_0)^2 + (c_1)^2 = \text{const.}$, and their canonical forms are parametrised by a closed half-line of real constants. On the other hand, in the hyperbolic case the structure is richer because equivalent systems correspond to connected branches of the hyperbolas $\Gamma_H = (c_0)^2 - (c_1)^2 = \text{const.}$; two connected components for $\Gamma_H > 0$, one for $\Gamma_H < 0$, and three for $\Gamma_H = 0$. Thus canonical forms of hyperbolic systems are parametrised by a real line of constants (the value of Γ_H) and by a discrete invariant $\varepsilon = \pm 1$ (if $\Gamma_H > 0$ or $\Gamma_H = 0$ and $c_0 \neq 0$). We also obtained a similar characterisation of flat and strongly flat elliptic/hyperbolic submanifolds; the derived classification of those strongly flat submanifolds is similar to the one of their parametrisations. The only notable difference shows up in some cases of hyperbolic submanifolds, where we can have two non-equivalent parametrisations of the two branches of an hyperbola whereas for the hyperbolic submanifold there is no distinction between the branches.

3.2 Classification of parabolic systems

We now turn to the classification of parabolic systems which is expected to be different from that of elliptic and hyperbolic systems because the allowed reparametrisations depend on 2 smooth functions (α, β), see Proposition 2.2. As in the elliptic/hyperbolic case, we introduce the structure functions $(\mu_0, \mu_1, \gamma_0, \gamma_1)$ uniquely defined for any P-frame (A, B) by $[A, B] = \mu_0 A + \mu_1 B$, and $C = \gamma_0 A + \gamma_1 B$. By a direct computation, we see that under reparametrisations those structure functions are transformed by

$$\tilde{\gamma}_0 = \tfrac{1}{\beta^2}\left(\gamma_0 - 2\alpha\gamma_1 - \alpha^2\right), \qquad \tilde{\gamma}_1 = \tfrac{1}{\beta}\left(\gamma_1 + \alpha\right), \tag{2.23}$$

$$\begin{aligned} \tilde{\mu}_0 &= \beta\mu_0 - 2\alpha \mathrm{L}_A(\beta) + 2\beta \mathrm{L}_A(\alpha) - 2\mathrm{L}_B(\beta) - 2\alpha\left(\mathrm{L}_A(\beta) + \beta\mu_1\right), \\ \tilde{\mu}_1 &= \beta^2\mu_1 + \beta \mathrm{L}_A(\beta). \end{aligned} \tag{2.24}$$

There are two main questions that we will answer. First, when does a commutative P-frame $(\tilde{A}, \tilde{B})$ exist (i.e. $\tilde{\mu}_0 = \tilde{\mu}_1 = 0$)? Second, provided that a P-frame (A, B) has been normalised, how can we additionally simplify C? Contrary to the elliptic and hyperbolic cases the answer to the first question is always positive without any additional assumption, as assured by the next result.

Proposition 2.6 (Existence of a commutative P-frame)**.**

(i) For any P-frame (A, B) there exists a reparametrisation (α, β) such that $(\tilde{A}, \tilde{B})$ is a commutative P-frame.

(ii) If (A, B) is a commutative P-frame, then $(\tilde{A}, \tilde{B})$ is also a commutative P-frame if and only if the reparametrisation (α, β) satisfies

$$\mathrm{L}_A(\beta) = 0 \quad \text{and} \quad \frac{1}{\beta}\mathrm{L}_B(\beta) = \mathrm{L}_A(\alpha). \tag{2.25}$$

Proof.

(i) Consider a P-frame (A, B) whose structure functions are (μ_0, μ_1). Apply a reparametrisation (α, β), $\beta \neq 0$, given by a solution of the following equations

$$\mathrm{L}_A(\beta) = -\beta\mu_1, \quad \text{and} \quad \mathrm{L}_A(\alpha) + \alpha\mu_1 = \frac{1}{2\beta}\left(2\mathrm{L}_B(\beta) - \beta\mu_0\right).$$

Notice that to ensure $\beta \neq 0$, we may actually solve $\mathrm{L}_A(\ln\beta) = -\mu_1$. Then, formula (2.24) implies that the structure functions of the new P-frame $(\tilde{A}, \tilde{B})$ satisfy $\tilde{\mu}_0 = \tilde{\mu}_1 = 0$. Therefore, $(\tilde{A}, \tilde{B})$ is a commutative P-frame.

(ii) Using relation (2.24) with $\mu_i = \tilde{\mu}_i = 0$ (for $i = 0, 1$), we see that all reparametrisations (α, β) have to satisfy relation (2.25). Conversely, if (A, B) is a commutative P-frame ($\mu_0 = \mu_1 = 0$) and (α, β) is any solution of (2.25) (with $\beta \neq 0$), then by (2.24) we see that $\tilde{\mu}_0 = \tilde{\mu}_1 = 0$.

■

Moreover, statement *(i)* of the above proposition gives the following prenormal forms for parabolic systems Ξ_P.

Corollary 2.4 (Prenormal forms of Ξ_P). *A parabolic system Ξ_P is always feedback equivalent to the following prenormal forms:*

$$\Xi'_P : \begin{cases} \dot{z} &= w^2 + c_0(x) \\ \dot{y} &= w + c_1(x) \end{cases}, \qquad \Xi''_P : \begin{cases} \dot{z} &= w^2 + b(x)w + \Gamma(x) \\ \dot{y} &= w \end{cases},$$

whose structure functions are $(\mu'_0, \mu'_1, \gamma'_0, \gamma'_1) = (0, 0, c_0, c_1)$, *and* $(\mu''_0, \mu''_1, \gamma''_0, \gamma''_1) = \left(\frac{\partial b}{\partial z}, 0, \Gamma, 0\right)$, *respectively.*

Remark. Since any parabolic system can be brought into Ξ'_P (and into Ξ''_P), it follows that all parabolic systems are parametrised by (roughly) two functions of two variables (c_0 and c_1, or, equivalently, b and Γ). This is in contrast with elliptic/hyperbolic systems Ξ_{EH} parametrised by three functions of two variables (compare Proposition 2.3). ◆

Proof. Apply to Ξ_P a reparametrisation (α, β) transforming its P-frame into a commutative P-frame $(\tilde{A}, \tilde{B})$ and introduce coordinates (z, y) such that $\tilde{A} = \frac{\partial}{\partial z}$ and $\tilde{B} = \frac{\partial}{\partial y}$. In this system of coordinates, Ξ_P takes the form Ξ'_P. Then apply to Ξ'_P the reparametrisation $\tilde{w} = w + c_1(x)$ to obtain the form Ξ''_P with $b = -2c_1$ and $\Gamma = c_0 + (c_1)^2$. The computation of structure functions is straightforward. ■

Notice that the normal forms Ξ'_P and Ξ''_P are related by the reparametrisation $\tilde{w} = w + c_1(x)$. The function Γ will be of special importance in the remaining part of this section, and in any P-frame (A, B) we define it by setting $\Gamma = \gamma_0 + (\gamma_1)^2$. Clearly, diffeomorphisms act on Γ by conjugation and reparametrisations (α, β) act by $\beta^2\tilde{\Gamma} = \Gamma$ (as it can be computed from formula (2.23)).

The remaining part of this section shows how to additionally normalise Ξ'_P and Ξ''_P while preserving the commutativity of the P-frame (A, B). Although there always exists a commutative quadratic frame (A, B), its explicit construction can be complicated, as it requires to solve a set of first order PDEs. For this reason, we will state our results for a general P-frame (A, B).

Theorem 2.4 (Normalisation of parabolic systems)**.** *Let $\Xi_P = (A, B, C)$ be a parabolic control system with structure functions $(\mu_0, \mu_1, \gamma_0, \gamma_1)$ defined by $[A, B] = \mu_0 A + \mu_1 B$ and $C = \gamma_0 A + \gamma_1 B$. Then the following statements hold.*

(i) *Ξ_P is locally feedback equivalent to Ξ'_P with $c_1 \equiv 0$ if and only if*

$$\mathrm{L}_A^2(\gamma_1) + \gamma_1\left(\mathrm{L}_A(\mu_1) - (\mu_1)^2\right) = \frac{\mu_0\mu_1}{2} + \frac{1}{2}\mathrm{L}_A(\mu_0) + \mathrm{L}_B(\mu_1). \tag{2.26}$$

(ii) *Ξ_P is locally feedback equivalent to Ξ'_P with $(c_0, c_1) \in \mathbb{R}^2$, satisfying $c_0 + (c_1)^2 \in \mathbb{R}^*$, if and only if $\Gamma \neq 0$ and it holds*

$$\mathrm{L}_A(\Gamma) + 2\mu_1\Gamma = 0, \quad \textit{and} \quad \mathrm{L}_B(\Gamma) + 2\Gamma\mathrm{L}_A(\gamma_1) = \Gamma\mu_0 - 2\Gamma\gamma_1\mu_1. \tag{2.27}$$

Moreover, in this case we can always normalise $c_0 = \pm 1$ and $c_1 = 0$.

(iii) *Ξ_P is locally feedback equivalent to Ξ'_P with $c_0 \equiv c_1 \equiv 0$ if and only if (2.26) holds and, additionally, $\Gamma \equiv 0$.*

Notice that statements *(i)*, *(ii)*, and *(iii)* of the above theorem characterise, respectively, the following normal forms

$$\Xi'''_P : \begin{cases} \dot{z} &= w^2 + c_0(x) \\ \dot{y} &= w \end{cases}, \qquad \Xi^{\pm}_P : \begin{cases} \dot{z} &= w^2 \pm 1 \\ \dot{y} &= w \end{cases}, \quad \text{and} \quad \Xi^0_P : \begin{cases} \dot{z} &= w^2 \\ \dot{y} &= w \end{cases}.$$

Equivalent statements can be made to obtain special structures of the second prenormal form Ξ''_P, e.g. the normal form Ξ'''_P corresponds to Ξ''_P with $b \equiv 0$ and thus describes the intersection of the prenormal forms Ξ'_P and Ξ''_P. The forms $\Xi^{\pm}_P$ and Ξ^0_P are then special cases of Ξ'''_P where c_0 is constant. The difference between the normal form $\Xi^{\pm}_P$ and Ξ^0_P lies in the existence or not of an equilibrium point: the control $w = 0$ defines an equilibrium of Ξ^0_P while there are no equilibria for $\Xi^{\pm}_P$.

Proof.

(i) **Sufficiency.** Consider a parabolic system $\Xi_P = (A, B, C)$ and assume that relation (2.26) holds. Using Proposition 2.6, apply (α, β) such that $(\tilde{A}, \tilde{B})$ is a commutative P-frame. For this new P-frame, condition (2.26) now reads $\mathrm{L}^2_{\tilde{A}}(\tilde{\gamma}_1) = 0$. Introduce the new reparametrisation $\alpha = -\tilde{\gamma}_1$ and β defined by $\mathrm{L}_{\tilde{A}}(\beta) = 0$ and $\mathrm{L}_{\tilde{B}}(\beta) = \beta\mathrm{L}_{\tilde{A}}(\alpha)$. The equations, rewritten for $\ln(\beta)$, admit solutions since the integrability condition, $\mathrm{L}^2_{\tilde{A}}(\alpha) = 0$ is fulfilled. The constructed reparametrisation (α, β) satisfies (2.25) and thus preserves the commutativity of $(\tilde{A}, \tilde{B})$. By applying (α, β) to $(\tilde{A}, \tilde{B}, \tilde{C})$, we obtain $(\hat{A}, \hat{B}, \hat{C})$ satisfying $\left[\hat{A}, \hat{B}\right] = 0$ and $\hat{\gamma}_1 = \frac{\tilde{\gamma}_1 + \alpha}{\beta} = 0$, by relation (2.23). By adopting coordinates (z, y) such that $(\hat{A}, \hat{B}) = \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial y}\right)$ we obtain the system Ξ'_P with $c_1 = \hat{\gamma}_1 = 0$.

(i) **Necessity.** Assume that $\Xi_P = (A, B, C)$ is feedback equivalent to $\tilde{\Xi}_P$ of the form Ξ'_P with $c_1 \equiv 0$, via ϕ and (α, β). Then for $\tilde{\Xi}_P$ we have $\tilde{\mu}_0 = \tilde{\mu}_1 = \tilde{\gamma}_1 = 0$. First, by (2.23) we obtain $\alpha = -\gamma_1$ and by (2.24) we obtain the following relations

$$\begin{cases} \frac{1}{\beta}\mathrm{L}_A(\beta) &= -\mu_1 \\ \frac{1}{\beta}\mathrm{L}_B(\beta) &= \frac{1}{2}\left(\mu_0 - 2\gamma_1\mu_1 - 2\mathrm{L}_A(\gamma_1)\right) \end{cases}.$$

Therefore, by computing the integrability condition of the latter system, we have

$$\mathrm{L}_A\left(\mathrm{L}_B\left(\ln\beta\right)\right)-\mathrm{L}_B\left(\mathrm{L}_A\left(\ln\beta\right)\right)=\mathrm{L}_{[A,B]}\left(\ln\beta\right)=\mu_0\mathrm{L}_A\left(\ln\beta\right)+\mu_1\mathrm{L}_B\left(\ln\beta\right),$$

which implies that

$$\frac{1}{2}\mathrm{L}_A\left(\mu_0\right)-\gamma_1\mathrm{L}_A\left(\mu_1\right)-\mathrm{L}_A^2\left(\gamma_1\right)\,-\mu_1\mathrm{L}_A\left(\gamma_1\right)+\mathrm{L}_B\left(\mu_1\right)=\\ -\frac{1}{2}\mu_0\mu_1-\gamma_1(\mu_1)^2-\mu_1\mathrm{L}_A\left(\gamma_1\right).$$

Hence, we conclude relation (2.26).

(ii) **Sufficiency.** Consider a parabolic system $\Xi_P=(A,B,C)$ and assume that $\Gamma=\gamma_0+(\gamma_1)^2\neq 0$ and that equation (2.27) holds. Using Proposition 2.6 apply a reparametrisation (α,β) such that $(\tilde A,\tilde B)$ is a commutative P-frame, and thus, for this new P-frame, condition (2.27) now reads

$$\mathrm{L}_{\tilde A}\left(\tilde\Gamma\right)=0\quad\text{ and }\quad\mathrm{L}_{\tilde B}\left(\tilde\Gamma\right)+2\tilde\Gamma\mathrm{L}_{\tilde A}\left(\tilde\gamma_1\right)=0.$$

Derivating the second relation along $\tilde A$ and using the commutativity of the Lie derivatives $\mathrm{L}_{\tilde A}()$ and $\mathrm{L}_{\tilde B}()$, i.e. $\mathrm{L}_{\tilde A}\left(\mathrm{L}_{\tilde B}\left(\cdot\right)\right)=\mathrm{L}_{\tilde B}\left(\mathrm{L}_{\tilde A}\left(\cdot\right)\right)$, together with $\tilde\Gamma\neq 0$ (recall that $\beta^2\tilde\Gamma=\Gamma$), one can see that $\mathrm{L}_{\tilde A}^2\left(\tilde\gamma_1\right)\,=0$. Introduce a new reparametrisation (α,β) defined by $\alpha=-\tilde\gamma_1$, $\mathrm{L}_{\tilde A}\left(\beta\right)=0$ and $\mathrm{L}_{\tilde B}\left(\beta\right)=\beta\mathrm{L}_{\tilde A}\left(\alpha\right)$. The equations, rewritten for $\ln(\beta)$, admit solutions since the integrability condition, $\mathrm{L}_{\tilde A}^2\left(\alpha\right)\,=0$, is fulfilled. The so constructed reparametrisation satisfies (2.25) and gives a commutative P-frame $(\hat A,\hat B)$, for which we obtain $\hat\mu_0=\hat\mu_1=\hat\gamma_1=0$. Therefore, we have $\hat\Gamma=\hat\gamma_0$ and condition (2.27) now reads $\mathrm{L}_{\hat A}\left(\hat\Gamma\right)=\mathrm{L}_{\hat B}\left(\hat\Gamma\right)=0$, implying that $\hat\Gamma$ is constant (we still have $\hat\Gamma\neq 0$). Introduce coordinates $(\hat z,\hat y)$ such that $\hat A=\frac{\partial}{\partial\hat z}$ and $\hat B=\frac{\partial}{\partial\hat y}$, in which the system takes the form

$$\begin{cases}\dot{\hat z}&=\hat w^2+c_0\\ \dot{\hat y}&=\hat w\end{cases},$$

with $c_0\in\mathbb{R}^*$. Finally, defining new coordinates (z,y) and reparametrising by, respectively, $z=\frac{\hat z}{|c_0|}$, $y=\frac{\hat y}{\sqrt{|c_0|}}$, and $w=\frac{\hat w}{\sqrt{|c_0|}}$, yields the normal form $\Xi_P^{\pm}$.

(ii) **Necessity.** Assume that Ξ_P, whose structure functions are $(\mu_0,\mu_1,\gamma_0,\gamma_1)$ and $\Gamma=\gamma_0+\gamma_1^2$, is feedback equivalent, via ϕ and (α,β), to $\tilde\Xi_P$ of the form Ξ_P' with $(c_0,c_1)\in\mathbb{R}^2$ satisfying $c_0+c_1^2\neq 0$. For $\tilde\Xi_P$ we have $\tilde\mu_0=\tilde\mu_1=0$ and $\tilde\gamma_0=c_0$, $\tilde\gamma_1=c_1$, hence $\tilde\Gamma=c_0+c_1^2\neq 0$ implying $\Gamma\neq 0$ since $\beta^2\tilde\Gamma=\Gamma$. By (2.24) we obtain the following relations

$$\begin{cases}\frac{1}{\beta}\mathrm{L}_A\left(\beta\right)&=-\mu_1\\ \frac{1}{\beta}\mathrm{L}_B\left(\beta\right)&=\frac{1}{2}\left(\mu_0-2\gamma_1\mu_1-2\mathrm{L}_A\left(\gamma_1\right)\right)\end{cases},$$

and differentiating $\Gamma=\beta^2\tilde\Gamma$ along A we deduce

$$\mathrm{L}_A\left(\Gamma\right)=\mathrm{L}_A\left(\beta^2\tilde\Gamma\right)=2\tilde\Gamma\beta\mathrm{L}_A\left(\beta\right)=-2\tilde\Gamma\beta^2\mu_1=-2\mu_1\Gamma,$$

giving the first relation of (2.27). A similar computation, by taking the derivative of $\Gamma=\beta^2\tilde\Gamma$ along B, implies the second relation of (2.27).

(iii) The proof of that statement is a special case of the proof of statement *(i)* with the additional condition $\Gamma \equiv 0$.

(iii) **Sufficiency.** Using the proof of the sufficiency of statement *(i)*, we bring the system Ξ_P into Ξ_P'''. For this form we have $\Gamma = c_0(x)$, hence $c_0(x) \equiv 0$ by assumption and we obtain the normal form Ξ_P^0.

(iii) **Necessity.** Assume that Ξ_P, whose structure functions are $(\mu_0, \mu_1, \gamma_0, \gamma_1)$ and $\Gamma = \gamma_0 + \gamma_1^2$, is feedback equivalent, via ϕ and (α, β), to $\tilde{\Xi}_P$ of the form Ξ_P^0 (which is, actually, Ξ_P' with $c_0 \equiv c_1 \equiv 0$). For that system we have $\tilde{\mu}_0 = \tilde{\mu}_1 = 0$ and $\tilde{\Gamma} \equiv 0$ and since Γ is transformed under (α, β) by $\beta^2\tilde{\Gamma} = \Gamma$, we see the necessity of $\Gamma \equiv 0$. The necessity of (2.26) is deduced from the necessity part of statement *(i)*.

■

Observe that statement *(ii)* does not explicitly require condition (2.26), while the normal form $\Xi_P^{\pm}$ satisfies $c_1 \equiv 0$ and hence that condition has to be hidden in (2.27). Indeed, this can be observed by differentiating Γ along $[A, B]$ and using the constraint (2.27), which after a short computation gives condition (2.26).

Remark (Interpretation of the conditions). We now give a tangible interpretation of our conditions. To this end, consider the system Ξ_P' for which we have $\mu_0 = \mu_1 = 0$, $\gamma_0 = c_0(x)$, $\gamma_1 = c_1(x)$ and thus $\Gamma(x) = c_0(x) + (c_1(x))^2$. First, condition (2.26) implies $\frac{\partial^2 c_1}{\partial z^2} = 0$, that is, $c_1(x)$ is affine with respect to z, namely, $c_1 = c_1^0(y)z + c_1^1(y)$, and thus, $\Gamma(x)$ is given by $c_0(x) + (c_1^0(y)z + c_1^1(y))^2$. This means that if a system Ξ_P' is feedback equivalent to Ξ_P''', then it is parametrised by 3 smooth functions, two of them being functions of y only, and it has the following form

$$\begin{cases} \dot{z} &= w^2 + \Gamma(x) - (c_1^0(y)z + c_1^1(y))^2 \\ \dot{y} &= w + c_1^0(y)z + c_1^1(y) \end{cases} .$$

By additionally applying the first equation of (2.27), we obtain $\frac{\partial c_0}{\partial z} + 2c_1^0(y)^2 z + 2c_1^0(y)c_1^1(y) = 0$ and thus $c_0(x)$ is a polynomial of degree 2 in z, related to $c_1(x)$ by $c_0(x) = -(c_1(x))^2 + c_2(y)$, for an arbitrary smooth function $c_2(y)$. We now have $\Gamma = \Gamma(y) = c_2(y)$ and we use the second equation of (2.27). We thus obtain $\Gamma(y) = G \exp\left(-2\int c_1^1(y)\,\mathrm{d}y\right)$, with $G \in \mathbb{R}$. To summarise, any system Ξ_P' satisfying (2.26) and (2.27) is parametrised by two arbitrary smooth functions of y and a constant $G \in \mathbb{R}$ and is expressed by the form

$$\begin{cases} \dot{z} &= w^2 + \Gamma(y) - (c_1^0(y)z + c_1^1(y))^2 \\ \dot{y} &= w + c_1^0(y)z + c_1^1(y) \end{cases} ,$$

where $\Gamma(y) = G \exp\left(-2\int c_1^1(y)\,\mathrm{d}y\right)$. Finally, Ξ_P' (satisfying (2.26) and (2.27)) is feedback equivalent to Ξ_P^0 if and only if $G = 0$, and is feedback equivalent to Ξ_P^+, respectively to Ξ_P^-, if $G > 0$, resp. $G < 0$. The distinction between the three normal forms comes from the sign of Γ, which is thus a discrete invariant (and that sign is dictated by the value of the constant G) ◆

Adapting the notation of Lemma 2.1, recall that a parabolic submanifold $\mathcal{S}_P$ is given by an equation of the form $\dot{z} = a(x)\dot{y}^2 + b(x)\dot{y} + c(x)$, with $a \neq 0$, and is then represented by the triple of functions (a, b, c). A parametrisation of $\mathcal{S}_P$ is given by a system $\Xi_{\mathcal{S}_P} = (A, B, C)$ with $A = a\frac{\partial}{\partial z}$, $B = b\frac{\partial}{\partial z} + \frac{\partial}{\partial y}$, and $C = c\frac{\partial}{\partial z}$. The following corollary gives normal forms of $\mathcal{S}_P$ and describes the underlying geometry.

Corollary 2.5 (Normalisations of parabolic submanifolds)**.** *Consider a parabolic submanifold $\mathcal{S}_P = (a, b, c)$ and define the following vector fields $A = a\frac{\partial}{\partial z}$, $B = b\frac{\partial}{\partial z} + \frac{\partial}{\partial y}$, and $C = c\frac{\partial}{\partial z}$. Then the following statements hold.*

(i) $\mathcal{S}_P$ is equivalent to $\mathcal{S}_P''' = (1, 0, c(x))$, if and only if $[A, [A, B]] = 0$ or, equivalently,

$$\frac{\partial}{\partial z}\left[\frac{1}{a}\left(a\frac{\partial b}{\partial z} - b\frac{\partial a}{\partial z} - \frac{\partial a}{\partial y}\right)\right] = 0. \tag{2.28}$$

(ii) $\mathcal{S}_P$ is equivalent to $\mathcal{S}_P^{\pm} = (1, 0, \pm 1)$, if and only if $C(\cdot) \neq 0$, $[A, C] = 0$ and $[B, C] = 0$ or, equivalently,

$$c \neq 0, \quad \frac{\partial}{\partial z}\left(\frac{c}{a}\right) = 0, \quad c\frac{\partial b}{\partial z} - b\frac{\partial c}{\partial z} - \frac{\partial c}{\partial y} = 0. \tag{2.29}$$

(iii) $\mathcal{S}_P$ is equivalent to $\mathcal{S}_P^0 = (1, 0, 0)$, if and only if $[A, [A, B]] = 0$ and $C = 0$ or, equivalently, $c \equiv 0$ and (2.28) holds.

Proof. Each statement is a special case of the previous theorem with the following structure functions $\mu_0 = \frac{1}{a}\left(a\frac{\partial b}{\partial z} - b\frac{\partial a}{\partial z} - \frac{\partial a}{\partial y}\right)$, $\gamma_0 = \frac{c}{a}$, $\mu_1 = \gamma_1 = 0$, and $\Gamma = \frac{c}{a}$. It is then straightforward to deduce condition (2.28) from (2.26), and condition (2.29) from (2.27). Then from the explicit form of the vector fields $A = a\frac{\partial}{\partial z}$, $B = b\frac{\partial}{\partial z} + \frac{\partial}{\partial y}$, and $C = c\frac{\partial}{\partial z}$ it is easy to deduce that (2.28) and (2.29) are equivalent to the corresponding commutativity conditions. ■

Constant curvature parabolic systems For elliptic and hyperbolic systems, we saw that the curvature (it was the Gaussian curvature of a well-defined metric there) plays a crucial role in the existence of a commutative frame. In the parabolic case, the existence of a commutative frame is guaranteed, thus we want to understand what it means for parabolic systems to have a vanishing *curvature*. Since parabolic systems correspond to degenerated metric tensor, we need some generalisation of the Riemannian curvature. It turns out that the notion of curvature defined by Agrachev in [Agr98] coincides (when $C \equiv 0$) with the Gaussian curvature $\mathfrak{g}_{\pm}$ for the elliptic and hyperbolic systems, as defined in (2.17). Thus it seems natural to use the curvature notion [Agr98] as curvature of parabolic systems.

In what follows, we compute the curvature[†] of the prenormal system Ξ'_P by studying its prolongation Σ'_P given by vector fields

$$f = \begin{pmatrix} w^2 + c_0(x) \\ w + c_1(x) \\ 0 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We follow [AS13, Chapter 23] to compute the curvature κ of Σ'_P which is, by construction, the curvature of Ξ'_P (see the conclusion of [AS13, Chapter 23]). The first

[†]The computations have been done by hand and verified by the symbolic computing environment Maple.

step is to compute, using the Hamiltonian, the singular control of Σ'_P, uniquely given by

$$u^s = -2\frac{\partial c_1}{\partial z}w^2 + \left(\frac{\partial c_0}{\partial z} - \frac{\partial c_1}{\partial y}\right) w + \frac{1}{2}\frac{\partial c_0}{\partial y}.$$

We apply to Σ'_P the feedback transformation $f \mapsto f^s = f + u^s g$ and after this transformation the singular control is zero or, equivalently, we have

$$[f^s, [f^s, g]] = -k_1 g - k_2 [f^s, g],$$

for some smooth functions k_1 and k_2. Finally, the curvature is given by the formula

$$\kappa = k_1 - \frac{1}{4}(k_2)^2 - \frac{1}{2}\mathrm{L}_{f^s}(k_2)$$

and leads to the following results. Notice that, contrary to the elliptic/hyperbolic case where the curvature, defined for classification purposes, was a true $(0,2)$-tensor on a surface, the notion of curvature of control systems used in this paragraph depends explicitly on the control w (in a polynomial way, see statement *(i)* below). Clearly, the curvature κ transforms under a diffeomorphism $\tilde{x} = \phi(x)$ by $\kappa = \phi^*\tilde{\kappa}$, therefore systems with constant curvature are of special importance and, below, we give a complete classification of parabolic systems with constant curvature.

Proposition 2.7 (Characterisation of flat parabolic systems)**.**

(i) *For any parabolic system Ξ'_P its curvature κ is a polynomial of degree 3 in w.*

(ii) *If the curvature κ is constant, then Ξ'_P is feedback equivalent to*

$$\Xi_P^\kappa : \left\{ \begin{array}{ll} \dot{z} & = w^2 + Ez - \left(\frac{1}{4}E^2 + \kappa\right) y^2 + Fy + G \\ \dot{y} & = w \end{array} \right. , \text{ around } x_0 = 0 \in \mathbb{R}^2,$$

and with $(E, F, G) \in \mathbb{R}^3$.

(iii) *Two systems Ξ_P^κ and $\tilde{\Xi}_P^{\tilde{\kappa}}$ of constant curvature, around $x_0 = 0 \in \mathbb{R}^2$, given by (κ, E, F, G) and $(\tilde{\kappa}, \tilde{E}, \tilde{F}, \tilde{G})$, respectively, are feedback equivalent if and only if*

$$\kappa = \tilde{\kappa}, \quad E = \tilde{E}, \quad \text{and} \quad (\tilde{F}, \tilde{G}) = (K\,F, K^2\,G), \quad K \in \mathbb{R}^*. \tag{2.30}$$

(iv) *If $\kappa \in \mathbb{R}$ and $E \in \mathbb{R}$ are fixed, then Ξ_P^κ is feedback equivalent around $0 \in \mathbb{R}^2$ to one of the following canonical forms:*

  (a) *$(\tilde{F}, \tilde{G}) = (0, 0)$ if and only if $(F, G) = (0, 0)$,*
  (b) *$(\tilde{F}, \tilde{G}) = (0, 1)$ if and only if $F = 0$ and $G > 0$,*
  (c) *$(\tilde{F}, \tilde{G}) = (0, -1)$ if and only if $F = 0$ and $G < 0$,*
  (d) *$(\tilde{F}, \tilde{G}) = (1, g)$, with $g \in \mathbb{R}$, if and only if $F \neq 0$*

Proof.

(i) A direct computation using f^s gives the expression of κ:

$$\kappa = \frac{\partial^2 c_1}{\partial z^2} w^3 - 3\left(\frac{1}{2}\left(\frac{\partial^2 c_0}{\partial z^2}\right) + \left(\frac{\partial c_1}{\partial z}\right)^2\right) w^2 + aw + b,$$

where a and b are functions of $x = (z, y)$.

(ii) If the curvature κ is constant, then the coefficients multiplying w^k, for $k = 1, 2, 3$, vanishes and the coefficient in front of w^0 is constant. First, the coefficient in front of w^3 gives $\frac{\partial^2 c_1}{\partial z^2} \equiv 0$, thus by Theorem 2.4 statement *(i)* the system Ξ'_P is feedback equivalent to Ξ'''_P (that is, Ξ'_P with $c_1 \equiv 0$). The structure functions of Ξ'''_P are $(0, 0, c_0, 0)$ so the expression of its curvature is

$$\kappa = -\frac{3}{2}\frac{\partial^2 c_0}{\partial z^2} w^2 - \frac{3}{2}\frac{\partial^2 c_0}{\partial z \partial y} w - \frac{1}{4}\left(\frac{\partial c_0}{\partial z}\right)^2 - \frac{1}{2}\frac{\partial^2 c_0}{\partial y^2} + \frac{c_0}{2}\frac{\partial^2 c_0}{\partial z^2}.$$

Second, we get $\frac{\partial c_0}{\partial z^2} \equiv 0$, thus $c_0 = c_0^1(y)z + c_0^2(y)$. Third, by $\frac{\partial^2 c_0}{\partial z \partial y} \equiv 0$, we obtain $c_0^1 = E \in \mathbb{R}$. And finally using $-\frac{1}{4}\left(\frac{\partial c_0}{\partial z}\right)^2 - \frac{1}{2}\frac{\partial^2 c_0}{\partial y^2} = \kappa \in \mathbb{R}$, equivalently, $2(c_0^2)''(y) = -4\kappa - E^2$, we obtain

$$c_0(z, y) = Ez - \left(\kappa + \frac{1}{4}E^2\right) y^2 + Fy + G,$$

with $(E, F, G) \in \mathbb{R}^3$.

(iii) We apply to Ξ^κ_P a reparametrisation $w = \beta\tilde{w} + \alpha$ that preserves both the commutativity of the P-frame $\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial y}\right)$ and $\gamma_1 = 0$. Therefore, $\alpha \equiv 0$ and $\beta \in \mathbb{R}^*$ (by Proposition 2.6); next, we introduce coordinates $(\tilde{z}, \tilde{y}) = \left(\frac{z}{\beta^2}, \frac{y}{\beta}\right)$ in which we get

$$\tilde{\Xi}^\kappa_P : \begin{cases} \dot{\tilde{z}} & = \tilde{w}^2 + E\tilde{z} - \left(\kappa + \frac{1}{4}E^2\right)\tilde{y}^2 + \frac{F}{\beta}\tilde{y} + \frac{G}{\beta^2} \\ \dot{y} & = \tilde{w} \end{cases}.$$

Now, by identification, we easily deduce relation (2.30) with $K = \frac{1}{\beta}$.

(iv) Using condition (2.30) we immediately conclude that $(\tilde{F}, \tilde{G}) = (0, 0)$ if and only if $(F, G) = (0, 0)$. If $F = 0$ and $G \neq 0$ then we have $\tilde{F} = 0$ and choosing $K = \frac{1}{\sqrt{|G|}}$ we get $\tilde{G} = \operatorname{sgn}(G) = \pm 1$. If $F \neq 0$ then choosing $K = \frac{1}{F}$ implies that $\tilde{F} = 1$ and $\tilde{G} = \frac{G}{F} = g \in \mathbb{R}$.

■

Notice that in the tuple (κ, E, F, G), we can always normalise F to 1 or 0. In the former case, for each fixed E and κ, we have a real line of systems parametrised by $g \in \mathbb{R}$. In the latter case, for each fixed E and κ, we have only three systems given by a discrete invariant, namely, the sign of G. In particular, observe that the class flat parabolic systems (i.e. those with zero curvature) contains the normal forms $\Xi^\pm_P$ and Ξ^0_P given as $\Xi^{\kappa=0}_P$ with $(E, F, G) = (0, 0, \pm 1)$ and $(E, F, G) = (0, 0, 0)$, respectively. From the last statement of the above proposition, notice that those normal forms, $\Xi^\pm_P$ and Ξ^0_P are the only canonical forms which do not depend on the point (z, y). Therefore, other systems with zero-curvature still depend on (z, y) and thus are not trivialisable (as called in [Ser09], when this problem has been investigated).

4 Conclusions and Perspectives

In this chapter, we studied control-nonlinear systems (evolving on a 2-dimensional manifold and with scalar control) under conic nonholonomic constraints. Those systems are treated as parametrisations of submanifolds of the tangent bundle, and thus all our results can equivalently be given for conic submanifolds of the tangent bundle under a certain type of equivalence. First, by studying the prolongation of those nonlinear systems, we gave a complete characterisation of conic submanifolds and we provided normal forms for systems satisfying that characterisation. Finally, working within the class of control-nonlinear systems subject to conic nonholonomic constraint, we exhibited several normal forms, in particular we highlighted a connection between the Gaussian curvature of a well-defined metric and the existence of a commutative frame for elliptic and hyperbolic systems. Normal forms include systems without functional parameters.

In future works we will extend our results to higher dimensional quadratic constraints, as well as we will study systems on 3-dimensional manifolds subject to a parabolic-elliptic or parabolic-hyperbolic constraint. Another interesting problem is to characterise nonlinear systems subject to any algebraic nonholonomic constraint. For instance, in order to generalise our results for parabolic systems, one can study polynomial systems, that is, systems subject to the nonholonomic constraint $\dot{z} - \sum_{i=0}^{d} a_i(x)\dot{y}^i = 0$.

2.A Resolution of equation (2.2)

Consider a smooth function $f = f(x, w)$ satisfying

$$\frac{\partial^3 f}{\partial w} = \tau(x)\frac{\partial f}{\partial w}.$$

We give an explicit form of the local solutions (around $w_0 \in \mathbb{R}$) of that equation. First, we find an expression for $\frac{\partial f}{\partial w}$ that will be then integrated to obtain the desired form of f. To this end, consider the following system of linear first order pdes:

$$\begin{pmatrix} \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial w} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \tau(x) & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

given for the functions $f_1 = \frac{\partial f}{\partial w}$ and $f_2 = \frac{\partial^2 f}{\partial w^2}$. Solutions of this system are expressed by the exponential of the matrix $\begin{pmatrix} 0 & w \\ \tau(x)w & 0 \end{pmatrix}$ given by the formula

$$\exp\begin{pmatrix} 0 & w \\ \tau(x)w & 0 \end{pmatrix} = \sum_{k=0}^{+\infty} \frac{w^{2k+1}\tau(x)^k}{(2k+1)!}\begin{pmatrix} 0 & 1 \\ \tau(x) & 0 \end{pmatrix} + \sum_{k=0}^{+\infty} \frac{w^{2k}\tau(x)^k}{(2k)!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

obtained by expressing the power series of the exponential and by regrouping the terms of odd and even degree. Denote $b(x) = f_1(x, w_0)$ and $a(x) = f_2(x, w_0)$ the initial conditions, thus we obtain

$$\frac{\partial f}{\partial w} = f_1(x, w) = a(x)\sum_{k=0}^{+\infty} \frac{(w - w_0)^{2k+1}}{(2k+1)!}\tau(x)^k + b(x)\sum_{k=0}^{+\infty} \frac{(w - w_0)^{2k}}{(2k)!}\tau(x)^k.$$

Integration of this expression yields

$$f(x, w) = a(x)\sum_{k=0}^{+\infty} \frac{(w - w_0)^{2k+2}}{(2k+2)!}\tau(x)^k + b(x)\sum_{k=0}^{+\infty} \frac{(w - w_0)^{2k+1}}{(2k+1)!}\tau(x)^k + c(x).$$

2.B Detailed computation for the proof of Theorem 2.3

We detail the computation made in the proof of Theorem 2.3.

2.B.1 Resolution of equation (2.9)

We show how to, locally around $0 \in \mathbb{R}^3$, solve the equation $\rho'' - 2\rho\rho' + \frac{4}{9}\rho^3 = 0$, where $\rho = \rho(x, w)$ and the derivatives are taken with respect to w. Take the new unknown function $R(x, w) = \exp\left(-\frac{2}{3}\int \rho(x, t)\,\mathrm{d}t\right)$ which satisfies $R(x, 0) \neq 0$ and

$$R' = -\frac{2}{3}\rho R, \quad R'' = -\frac{2}{3}R\left(\rho' - \frac{2}{3}\rho^2\right), \quad R''' = -\frac{2}{3}R\left(\rho'' - 2\rho\rho' + \frac{4}{9}\rho^3\right) = 0.$$

Thus, $R(x, w) = a(x)w^2 + b(x)w + c(x)$ yielding $\rho = -\frac{3}{2}\frac{R'}{R} = -\frac{3}{2}\frac{2aw+b}{aw^2+bw+c}$. Since $R(x, 0) \neq 0$, we have $c \neq 0$, thus taking $d(x) = \frac{a}{c}$ and $e(x) = \frac{b}{c}$ we obtain

$$\rho(x, w) = -\frac{3}{2}\frac{2d(x)w + e(x)}{d(x)w^2 + e(x)w + 1}.$$

2.B.2 Smooth form of h'

We integrate $h''(x,w) = a(x)(d(x)w^2 + e(x)w + 1)^{-3/2}$ with respect to w to obtain a smooth expression of $h'(x,w)$ around $0 \in \mathbb{R}^3$. Recall that we denote $p = p(x,w) = d(x)w^2 + e(x)w + 1$ and $\Delta = \Delta(x) = e(x)^2 - 4d(x)$. We have

$$h'(x,w) = a(x) \int \frac{1}{p(x,w)^{3/2}}\, dw = \frac{-2a(2dw+e)}{\Delta\sqrt{p}} + \bar{b}(x).$$

Since $h'(x,0) = b(x)$ is smooth, we have $\frac{-2ae}{\Delta} + \bar{b} = b(x)$, where $b(x)$ is a smooth function around 0. We can then derive a smooth closed form expression of $h'(x,w)$:

$$\begin{aligned}
h'(x,w) &= \frac{-2a(2dw+e)}{\Delta\sqrt{p}} + \frac{2ae}{\Delta} + b = \frac{-2a}{\Delta\sqrt{p}}\,(2dw + e - e\sqrt{p}) + b,\\
&= \frac{-2a}{\Delta\sqrt{p}(ew+2+2\sqrt{p})}\,(2dw + e - e\sqrt{p})\,(ew + 2 + 2\sqrt{p}) + b,\\
&= \frac{-2a}{\Delta\sqrt{p}(ew+2+2\sqrt{p})}\big(2dew^2 + 4dw + 4dw\sqrt{p} + e^2w + 2e\\
&\qquad\qquad +2e\sqrt{p} - e^2w\sqrt{p} - 2e\sqrt{p} - 2ep\big) + b,\\
&= \frac{-2a}{\Delta\sqrt{p}(ew+2+2\sqrt{p})}\big(w\sqrt{p}(4d-e^2) + 4dw + 2dew^2 + 2e\\
&\qquad\qquad +e^2w - 2edw^2 - 2e^2w - 2e\big) + b,\\
&= \frac{-2a}{\Delta\sqrt{p}(ew+2+2\sqrt{p})}\big(w\sqrt{p}(4d-e^2) + w(4d-e^2)\big) + b,\\
&= \frac{2aw}{\sqrt{p}(ew+2+2\sqrt{p})}\,(\sqrt{p}+1) + b.
\end{aligned}$$

2.C Detailed computation for the proof of Lemma 2.2

Denoting $c_E(x) = \cos(x)$, $c_H(w) = \cosh(w)$, $s_E(w) = \sin(w)$, and $s_H(w) = \sinh(w)$ and starting from system

$$\Xi_{EH} \,:\, \dot{x} = (A,B)\begin{pmatrix} c_{EH}(w)\\ s_{EH}(w)\end{pmatrix} + (A,B)\begin{pmatrix}\gamma_0\\ \gamma_1\end{pmatrix},$$

we apply a reparametrisation $w = \tilde{w} + \alpha(x)$:

$$\begin{aligned}
\dot{x} &= (A,B)R_{EH}(\pm\alpha)\begin{pmatrix} c_{EH}(\tilde{w})\\ s_{EH}(\tilde{w})\end{pmatrix} + (A,B)\begin{pmatrix}\gamma_0\\ \gamma_1\end{pmatrix},\\
&= (\tilde{A},\tilde{B})\begin{pmatrix} c_{EH}(\tilde{w})\\ s_{EH}(\tilde{w})\end{pmatrix} + (\tilde{A},\tilde{B})R_{EH}(\pm\alpha)^{-1}\begin{pmatrix}\gamma_0\\ \gamma_1\end{pmatrix}.
\end{aligned}$$

This yields $\big(\tilde{\gamma}_0, \tilde{\gamma}_1\big) = \big(\gamma_0, \gamma_1\big)\, R_{EH}(\pm\alpha)^{-T}$, and by definition of $R_{EH}(\alpha)$ we have,

$$R_E(\alpha)^{-T} = R_E(-\alpha)^t = R_E(\alpha), \quad \text{and} \quad R_H(-\alpha)^{-T} = R_H(-\alpha)^{-1} = R_H(\alpha).$$

Next, computing separately in the elliptic and hyperbolic cases, we obtain

$$\begin{gathered}
\big[\tilde{A},\tilde{B}\big] = (\mu_0 \mp \mathrm{L}_A(\alpha))A + (\mu_1 - \mathrm{L}_B(\alpha))B,\\
(A,B)R_E(\pm\alpha)\begin{pmatrix}\tilde{\mu}_0\\ \tilde{\mu}_1\end{pmatrix} = (A,B)\begin{pmatrix}\mu_0 \mp \mathrm{L}_A(\alpha)\\ \mu_1 - \mathrm{L}_B(\alpha)\end{pmatrix}.
\end{gathered}$$

Thus

$$\left(\tilde{\mu}_0, \tilde{\mu}_1\right) = \left(\mu_0 \mp \mathrm{L}_A\left(\alpha\right), \mu_1 - \mathrm{L}_B\left(\alpha\right)\right) R_E(\pm\alpha)^{-T}$$

and the relation (2.15) follows.

2.D Gaussian curvature for a metric given by a moving frame

Consider a 2-dimensional manifold $\mathcal{X}$ and two smooth vector fields A, and B satisfying $A \wedge B \neq 0$. Construct the (pseudo)-Riemanian metric $\mathsf{g}_\pm$ defined by

$$\mathsf{g}_\pm(A, A) = 1, \quad \mathsf{g}_\pm(B, B) = \pm 1, \quad \text{and} \quad \mathsf{g}_\pm(A, B) = 0.$$

We will give a formula for the Gaussian curvature of $\mathsf{g}_\pm$ in terms of the structure functions (μ_0, μ_1) uniquely defined by $[A, B] = \mu_0 A + \mu_1 B$. We will use the following formula for the covariant derivative

$$\nabla_{E_i} E_j = \frac{1}{2} \sum_k \left(\mathsf{g}_\pm([E_i, E_j], E_k) - \mathsf{g}_\pm([E_i, E_k], E_j) - \mathsf{g}_\pm([E_j, E_k], E_i)\right) E_k$$

for $E_i \in \{A, B\}$, and the following formula for the Gaussian curvature of a 2-dimensional manifold,

$$\kappa_\pm = \frac{\mathsf{g}_\pm\left((\nabla_B \nabla_A - \nabla_A \nabla_B + \nabla_{[A,B]})A, B\right)}{\det(\mathsf{g}_\pm)}.$$

Computing, we have

$$\nabla_A A = -\mu_0\, B, \quad \nabla_A B = \mu_0\, A, \quad \nabla_B A = \mp\mu_1\, B, \quad \nabla_B B = \pm\mu_1\, A.$$

Then we can deduce

$$\begin{aligned}
&\nabla_B \nabla_A A = -\mathrm{L}_B\left(\mu_0\right) B \mp \mu_0\mu_1 A, \quad \nabla_A \nabla_B A = \mp\mathrm{L}_A\left(\mu_1\right) B \mp \mu_0\mu_1 A, \\
&\nabla_{[A,B]} A = \mu_0 \nabla_A A + \mu_1 \nabla_B A = -(\mu_0)^2 B \mp (\mu_1)^2 B.
\end{aligned}$$

Thus

$$\begin{aligned}
\kappa_\pm &= \pm\mathsf{g}_\pm\left(\left(-\mathrm{L}_B\left(\mu_0\right) \pm \mathrm{L}_A\left(\mu_1\right) - (\mu_0)^2 \mp (\mu_1)^2\right) B, B\right), \\
&= -\mathrm{L}_B\left(\mu_0\right) \pm \mathrm{L}_A\left(\mu_1\right) - (\mu_0)^2 \mp (\mu_1)^2.
\end{aligned}$$

Chapter 3

Introduction to the equivalence problem of control systems with paraboloid nonholonomic constraints

In this chapter, we introduce the notations and the mathematical tools that we need for our characterisation and classification of paraboloid submanifolds in any dimension. The following Chapter 4 is dedicated to the study of the three dimensional case and Chapter 5 deals with the case of arbitrary dimension.

First, we recall some general facts on the problems that we are interested in. In the tangent bundle $T\mathcal{X}$, of a smooth n-dimensional manifold $\mathcal{X}$ (with $n \geq 2$ and equipped with local coordinates x), we consider a smooth $(2n-1)$-dimensional submanifold (a hypersurface)

$$\mathcal{S} = \{(x, \dot{x}) \in T\mathcal{X},\ S(x, \dot{x}) = 0\},$$

where $S : T\mathcal{X} \to \mathbb{R}$ satisfies $\operatorname{rk} \frac{\partial S}{\partial \dot{x}}(x, \dot{x}) = 1$ for all $(x, \dot{x}) \in \mathcal{S}$. Our purpose is to find necessary and sufficient conditions ensuring that $\mathcal{S}$ describes a quadric surface, i.e., we want to characterise the equivalence (under diffeomorphisms of $\mathcal{X}$ and multiplication by a nonvanishing function of $T\mathcal{X}$) of $\mathcal{S}$ to

$$\mathcal{S}_q = \left\{(x, \dot{x}) \in T\mathcal{X},\ \dot{x}^t \mathtt{g}(x)\dot{x} + 2\omega(x)\dot{x} + h(x) = 0\right\},$$

where $\mathtt{g}(x)$ is smooth symmetric matrix (representing a symmetric $(0,2)$-tensor), $\omega(x)$ is a smooth covector (representing a differential one-form), and $h(x)$ is a smooth scalar function. As in the previous chapter, we are interested in non-degenerate quadrics, that is non-empty sets $\mathcal{S}_q$ satisfying $\Delta_1 = \det\left(\begin{smallmatrix} \mathtt{g} & \omega^t \\ \omega & h \end{smallmatrix}\right) \neq 0$. The latter assumption implying that $\operatorname{rk} \mathtt{g}(x) \geq n-1$ everywhere.

Remark (Quadrics in dimension $n = 2$ and $n = 3$). When $\dim \mathcal{X} = 2$, then the quadrics are called conics, among which we find ellipses, hyperbolas, and parabolas (those were studied in the previous chapter). When $\dim \mathcal{X} = 3$, there are 5 of non-degenerate conics, namely: the ellipsoid, the one and two sheeted hyperboloids, the elliptic paraboloid and the hyperbolic paraboloid. The last two classes correspond to quadric surface for which $\mathtt{g}$ is degenerate, i.e. it locally satisfies $\operatorname{rk} \mathtt{g}(x) = 2$, they are studied in Chapter 4. ◆

In the next two chapters, we will exclusively study quadrics satisfying $\operatorname{rk} \mathsf{g}(x) = n-1$ in a neighbourhood. Therefore, the results of the following chapters generalise the results obtained in Chapter 2 for parabolic submanifolds (in particular, see statement *(i)* of Corollary 2.1 for a characterisation result and see Section 3.2 for classification results). The quadric surfaces $\mathcal{S}_q$ with $\operatorname{rk} \mathsf{g} = n-1$ are called *paraboloid* surfaces by analogy with the terminology in dimension $n = 3$ (see remark above). In a suitable coordinate system $x = (z, y)$, chosen such that $\ker \mathsf{g} = \operatorname{span}\left\{\frac{\partial}{\partial z}\right\}$, they are given by

$$\mathcal{S}_Q = \{(x, \dot{x}) \in T\mathcal{X},\ \dot{z} = \dot{y}^t Q(x)\dot{y} + 2b(x)\dot{y} + c(x)\},$$

where $Q(x)$ is a smooth $(n-1)$ by $(n-1)$ symmetric matrix of full rank, $b(x) = (b_1(x), \ldots, b_{n-1}(x))$ is smooth covector, and $c(x)$ is a smooth scalar function. We denote $(p, q)(x) \in \mathbb{N}^2$ the signature of $Q(x)$ where $p(x)$ (resp. $q(x)$) is the number of positive (resp. negative) eigenvalues at x and we have $p(x) + q(x) = n-1$ everywhere. By the diffeomorphism $\tilde{z} = -z$ we can always transform (p, q) into (q, p) thus we will use the convention $p \geq q$. As we assumed that $Q(x)$ has full rank, it follows that the signature (p, q) is constant in a neighbourhood. Moreover, by a straightforward calculation, we deduce that if two paraboloid surfaces $\mathcal{S}_Q$ and $\mathcal{S}_{\tilde{Q}}$ are equivalent then $(p, q) = (\tilde{p}, \tilde{q})$. Therefore, that signature (p, q) is a first invariant of our equivalence problem.

Definition 3.1 ((p,q)-parabolisable submanifold)**.** We say that a submanifold $\mathcal{S}$ is (p, q)-parabolisable if it is equivalent (in the sense of Definition 1.5 of Chapter 1) to $\mathcal{S}_Q$, with $\operatorname{sgn}(Q) = (p, q)$.

Recall that to any submanifold $\mathcal{S}$ we attach two parametrisations: $\Xi_{\mathcal{S}}$ (first prolongation) and $\Sigma_{\mathcal{S}}$ (second prolongation) given by

$$\Xi_{\mathcal{S}} \,:\, \dot{x} = F(x, w) \quad w \in \mathcal{W} \subset \mathbb{R}^m, \quad \text{and} \quad \Sigma_{\mathcal{S}} \,:\, \begin{cases} \dot{x} &= F(x, w) \\ \dot{w} &= u \end{cases} \quad u \in \mathbb{R}^m,$$

where $m = n-1$, and $F(x, w)$ is a smooth map satisfying $\operatorname{rk} \frac{\partial F}{\partial w}(x, w) = m$ and $S(x, F(x, w)) = 0$ for all w. Those prolongations are seen as control-nonlinear systems and control-affine systems, respectively. From Proposition 1.6 of Chapter 1, we know that the equivalence of submanifolds under the action of diffeomorphisms and nonvanishing functions is equivalent to the equivalence of the corresponding first and second prolongation under feedback transformations. Thus, the problem of characterising submanifolds $\mathcal{S}_Q$ will be replaced by that of characterising their second prolongations defined by

$$\Sigma_{\mathcal{S}_Q} \,:\, \begin{cases} \dot{z} &= w^t Q(x) w + b(x) w + c(x) \\ \dot{y} &= w \\ \dot{w} &= u \end{cases}.$$

The state (z, y, w) belongs to a smooth $(2m+1)$-dimensional manifold $\mathcal{M}$, and $u = (u_1, \ldots, u_m)$ is the control. The following definition defines a class of control-affine systems that turns out to be equivalent to the class given by $\Sigma_{\mathcal{S}_Q}$, see Lemma 3.1 below. The element of this new class are more useful in practical computations.

Definition 3.2 ((p,q)-parabolisable systems)**.** We say that a control-affine system Σ on a $(2m+1)$-dimensional manifold $\mathcal{M}$ and with m controls, is (p,q)-*parabolisable* if it is feedback equivalent to

$$\Sigma_{p,q} : \begin{cases} \dot{x} &= A(x) w^t \mathtt{I}_{p,q} w + B(x) w + C(x) \\ \dot{w} &= u \end{cases}, \quad (x,w) \in \mathcal{M},\ u \in \mathbb{R}^m,$$

where $\mathtt{I}_{p,q} = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}$, A, $B = (B_1, \dots, B_m)$, and C are smooth vector fields on $\mathcal{M}/_\sim$, where the equivalence relation is defined by the integrable distribution span $\left\{ \frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial w_m} \right\}$, satisfying $A \wedge B_1 \wedge \dots \wedge B_m \neq 0$.

The following lemma shows that the two classes of systems of the form $\Sigma_{\mathcal{S}_Q}$ and $\Sigma_{p,q}$ coincide up to feedback transformations.

Lemma 3.1. *A system $\Sigma_{p,q}$ is locally feedback equivalent to $\Sigma_{\mathcal{S}_Q}$ and, conversely, a system $\Sigma_{\mathcal{S}_Q}$ is locally feedback equivalent to $\Sigma_{p,q}$. Moreover, in the equivalences above,* $\mathrm{sgn}\,(Q) = (p,q)$.

Proof. Consider a control-affine system of the form $\Sigma_{p,q}$ and apply a local diffeomorphism $(z,y) = (z, y_1, \dots, y_m) = \phi(x)$ such that $\phi_* A = \frac{\partial}{\partial z}$. In that coordinate system we have

$$\Sigma_{p,q} : \begin{cases} \dot{z} &= w^t \mathtt{I}_{p,q} w + b(x) w + c(x) \\ \dot{y} &= \bar{B}(x) w + \bar{C}(x) \\ \dot{w} &= u \end{cases},$$

with (using again the symbols A, B, and C) $A = \frac{\partial}{\partial z}$, $(B_1, \dots, B_m) = \begin{pmatrix} b_1 & \dots & b_m \\ \bar{B}_1 & \dots & \bar{B}_m \end{pmatrix}$, and $C = \left(\begin{smallmatrix} c \\ \bar{C} \end{smallmatrix} \right)$. By assumption, we have $A \wedge B_1 \wedge \dots \wedge B_m \neq 0$ implying that $\det \bar{B} \neq 0$. Therefore, we introduce new coordinates $\tilde{w} = \bar{B} w + \bar{C}$ which yields $\dot{y} = \tilde{w}$ and

$$\dot{z} = \tilde{w}^t \tilde{Q}(x) \tilde{w} + \tilde{b}(x) \tilde{w} + \tilde{c}(x),$$

where $\tilde{Q} = \bar{B}^{-t} \mathtt{I}_{p,q} \bar{B}^{-1}$, $\tilde{b} = -2\bar{C}^t \tilde{Q} + b\bar{B}^{-1}$, and $\tilde{c} = \bar{C}^t \tilde{Q} \bar{C} - b\bar{B}^{-1}\bar{C} + c$. Thus, we effectively obtain a second prolongation $\Sigma_{\mathcal{S}_{\tilde{Q}}}$ of a paraboloid surface $\mathcal{S}_{\tilde{Q}}$. Clearly, the relation between $\tilde{Q}$ and $\mathtt{I}_{p,q}$ imply that $\mathrm{sgn}\left(\tilde{Q}\right) = \mathrm{sgn}\,(\mathtt{I}_{p,q}) = (p,q)$.

Conversely, consider a control-affine system of the form $\Sigma_{\mathcal{S}_Q}$. Taking a new coordinate $\bar{w}$ given by $w = D\bar{w}$, with D a constant diagonal matrix, we transform Q into $\bar{Q}(x) = D^t Q(x) D$. Hence by choosing D suitably we can ensure that the eigenvalues of $\bar{Q}$ are simple in a neighbourhood of x_0 (actually it is sufficient to obtain $\bar{Q}(x_0)$ with simple eigenvalues). Notice that $\bar{Q}$ has the same signature (p,q) as Q, thus $\bar{Q}$ posses p positive and q negative eigenvalues around x_0. Using a smooth version of Sylvester's law of inertia (given by the theorem in [Fre82]) we conclude that there (locally) exists $P(x)$ such that $P(x)^t \bar{Q}(x) P(x) = \mathtt{I}_{p,q}$. In the new coordinate system given by $\bar{w} = P(x)\tilde{w}$, the system $\Sigma_{\mathcal{S}_Q}$ takes the form $\Sigma_{p,q}$ (after applying a suitable feedback along the $\tilde{w}$-components) with $A = \frac{\partial}{\partial z}$, $B = \begin{pmatrix} b(x)DP(x) \\ DP(x) \end{pmatrix}$, and $C = \left(\begin{smallmatrix} c \\ 0 \end{smallmatrix} \right)$. ∎

This lemma justifies the following terminology. We call $\Sigma_{p,q}$ a (p,q)-*paraboloid* system, shortly a (p,q)-system. And, we will replace the problem of characterising paraboloid submanifolds $\mathcal{S}_Q$ by that of paraboloid systems $\Sigma_{p,q}$.

Preliminary results. In this paragraph, we introduce tools that will serve for the characterisation of (p,q)-systems. To this end, we consider a control-affine system $\Sigma = (f, g)$ with state $\xi \in \mathcal{M}$, a $(2m+1)$-dimensional manifold, and with m controls $u = (u_1, \dots, u_m)$, defined by

$$\Sigma \,:\, \dot{\xi} = f(\xi) + \sum_{i=1}^{m} g_i(\xi)\, u_i, \qquad (u_1, \dots, u_m) \in \mathbb{R}^m.$$

Any such system Σ is given by $m+1$ smooth vector fields: the drift f, and the m-tuple $(g_1, \dots, g_m)$. We define two distributions attached to the system Σ

$$\mathcal{D}^0 = \operatorname{span}\{g_1, \dots, g_m\} \quad \text{and} \quad \mathcal{D}^1 = \operatorname{span}\{g_1, \dots, g_m, \operatorname{ad}_f g_1, \dots, \operatorname{ad}_f g_m\}.$$

We say that the m-tuple $g := (g_1, \dots, g_m)$ is a *frame* of $\mathcal{D}^0$. Throughout the following chapters, we will assume that Σ satisfies

(A1) The distribution $\mathcal{D}^0$ is involutive and has constant rank m,

(A2) The distribution $\mathcal{D}^1$ has constant rank $2m$.

If $\mathcal{D}^0$ is involutive, then the distributions $\mathcal{D}^0$ and $\mathcal{D}^1$ are invariant under feedback transformations of the form $u = \alpha(\xi) + \beta(\xi)\tilde{u}$. Assumptions (A1) and (A2) are justified by the fact that they are satisfied by any second prolongation $\Sigma_{\mathcal{S}}$ of a submanifold $\mathcal{S}$. Under assumption (A2), the codistribution $\operatorname{ann}(\mathcal{D}^1)$ is of constant rank one and we define the map

$$\begin{aligned} \Omega \,:\, \operatorname{ann}\left(\mathcal{D}^1\right) \times \mathcal{D}^0 \times \mathcal{D}^0 &\longrightarrow \mathbb{R} \\ (\omega, g_i, g_j) &\longmapsto \Omega(\omega, g_i, g_j) = \omega\left([g_i, \operatorname{ad}_f g_j]\right), \end{aligned}$$

which will be of special importance in our work. We denote by Ω_ω the matrix representation of the application $\Omega(\omega, \cdot, \cdot)$ for a fixed $\omega \neq 0$ evaluated on a fixed frame g of $\mathcal{D}^0$. We emphasize the fact that Ω_ω plays the role of an invariant hessian matrix of f with respect to the directions of the fields g_i for $i = 1, \dots, m$. Indeed, when $\Sigma = (f, g)$ is the straightforward second prolongation of the submanifold $\mathcal{S} = \{\dot{z} = s(x, \dot{y})\}$, that is we have $f = s(x, w)\frac{\partial}{\partial z} + \sum_{i=1}^{m} w_i \frac{\partial}{\partial y_i}$, and $g_i = \frac{\partial}{\partial w_i}$, this matrix Ω_ω is, up to scaling by a non-vanishing function, the hessian of the function $s(x, w)$ with respect to $w = (w_1, \dots, w_m)$.

Lemma 3.2 (Properties of the application Ω_ω)**.** *Assume that (A1) and (A2) hold and fix a one-form ω such that* $\operatorname{span}\{\omega\} = \operatorname{ann}(\mathcal{D}^1)$. *The following statements hold:*

(i) *Ω_ω is a smooth symmetric $(0,2)$-tensor on $\mathcal{D}^0$;*

(ii) *For any $\tilde{\omega}$ satisfying* $\operatorname{span}\{\tilde{\omega}\} = \operatorname{ann}(\mathcal{D}^1)$, *we have $\Omega_{\tilde{\omega}} = \lambda\, \Omega_\omega$, where $\tilde{\omega} = \lambda\omega$;*

(iii) *Feedback transformations $f \mapsto \tilde{f} = f + g\alpha$ and $g \mapsto \tilde{g} = g\beta$ transform Ω_ω into*

$$\tilde{\Omega}_\omega = \beta^t \Omega_\omega \beta, \tag{3.1}$$

where $\tilde{\Omega}_\omega$ is the matrix of the application $\Omega(\omega, \cdot, \cdot)$ evaluated on the new frame $\tilde{g}$ of $\mathcal{D}^0$, i.e. $\tilde{\Omega}_\omega(\tilde{g}_i, \tilde{g}_j) = (\beta^t \Omega_\omega \beta)(g_i, g_j)$;

(iv) *For any (p,q)-paraboloid system $\Sigma_{p,q}$ we have* $\operatorname{sgn}(\Omega_\omega) = (p,q)$ *up to order.*

Proof.

(i) The computation is straightforward and is given here for completeness. For any two fields g_i and g_j of $\mathcal{D}^0$ we have

$$\Omega(\omega, g_j, g_i) = \omega\left([g_j, \mathrm{ad}_f g_i]\right) = \omega\left([g_i, \mathrm{ad}_f g_j] - [f, [g_i, g_j]]\right) = \Omega(\omega, g_i, g_j),$$

where we used the Jacobi identity and the involutivity of $\mathcal{D}^0$. Moreover,

$$\begin{aligned}\Omega(\omega, g_i + g_k, g_j) &= \omega\left([g_i + g_k, \mathrm{ad}_f g_j]\right) = \Omega(\omega, g_i, g_j) + \Omega(\omega, g_k, g_j),\\ \Omega(\omega, \lambda g_i, g_j) &= \omega\left([\lambda g_i, \mathrm{ad}_f g_j]\right) = \lambda\Omega(\omega, g_i, g_j),\end{aligned}$$

for any $\lambda \in C^\infty(\mathcal{M})$.

(ii) Under assumption (A2), the distribution $\mathcal{D}^1$ is of corank one and therefore, for any ω and $\tilde{\omega}$ such that $\mathrm{span}\,\{\omega\} = \mathrm{ann}\,(\mathcal{D}^1) = \mathrm{span}\,\{\tilde{\omega}\}$, we have $\omega = \lambda\,\tilde{\omega}$ with $\lambda(\cdot) \neq 0$ and the conclusion follows immediately.

(iii) By assumption (A1), it is clear that Ω is invariant under the transformation $f \mapsto f + g\alpha$. Let $g = (g_1, \ldots, g_m)$ be a frame of $\mathcal{D}^0$ and denote $\Omega_{i,j} = \Omega(\omega, g_i, g_j)$ for $i, j = 1, \ldots, m$, recall that Ω_ω is a symmetric matrix. Consider the transformation $\tilde{g} = g\beta$ for $\beta \in C^\infty(\mathcal{M}, GL_m(\mathbb{R}))$. For $i = 1, \ldots, m$, we have $\tilde{g}_i = \sum_{k=1}^m g_k \beta_i^k$ giving

$$\begin{aligned}\tilde{\Omega}_{i,j} &= \sum_{k,l} \omega\left([\tilde{g}_i, \mathrm{ad}_f \tilde{g}_j]\right) = \sum_{k,l} \omega\left(\left[g_k\beta_i^k, \mathrm{ad}_f(g_l\beta_j^l)\right]\right)\\ &= \sum_{k,l} \omega\left(\beta_i^k\left[g_k, \mathrm{ad}_f g_l\right]\beta_j^l\right) = \sum_{k,l} \beta_i^k \Omega_{k,l}\beta_j^l,\end{aligned}$$

which exactly translates into $\tilde{\Omega}_\omega = \beta^t \Omega_\omega \beta$.

(iv) For a (p,q)-paraboloid system $\Sigma_{p,q} = (f, g)$, we have $f = A(x)w^t \mathtt{I}_{p,q} w + B(x)w + C(x)$ and $g_i = \frac{\partial}{\partial w_i}$; hence, $\mathrm{ad}_f g_i = -2A\mathtt{I}_i^i w_i - B_i$ (no summation convention). Thus, we have the distributions $\mathcal{D}^0 = \mathrm{span}\left\{\frac{\partial}{\partial w_1}, \ldots, \frac{\partial}{\partial w_m}\right\}$ and $\mathcal{D}^1 = \mathcal{D}^0 + \mathrm{span}\,\{2AI_i^i w_i + B_i,\ i = 1, \ldots, m\}$. So by a straightforward computation we have, modulo $\mathcal{D}^0$,

$$[g_i, \mathrm{ad}_f g_i] = -2A\mathtt{I}_i^i \qquad \text{and} \qquad [g_i, \mathrm{ad}_f g_j] = 0 \quad \text{for} \quad i \neq j.$$

Thus, for $\omega \in \mathrm{ann}\,(\mathcal{D}^1)$ we have $\Omega_\omega = -2\omega(A)\mathtt{I}_{p,q}$, where $\omega(A) := \omega(A + 0 \cdot \frac{\partial}{\partial w})$. Since $A \wedge B_1 \wedge \ldots \wedge B_m \neq 0$ we deduce that $A \notin \mathcal{D}^1$, i.e. $\omega(A) \neq 0$, and the conclusion follows. If $\mathrm{sgn}\,(\Omega_\omega) = (p,q)$, then $\mathrm{sgn}\,(\Omega_{-\omega}) = (q,p)$, which justifies the statement about the order.

■

The symmetric bilinear form Ω_ω plays a crucial role in our work. By statement *(iv)* of the previous lemma the (p,q)-paraboloid systems $\Sigma_{p,q}$ have Ω_ω of constant signature (p,q). By statement *(iii)*, we conclude that the signature is invariant under feedback transformations, thus it is necessary to add, on general control-affine systems Σ, the assumption

(A3) $\operatorname{sgn}(\Omega_\omega) = (p, q)$ is constant and satisfies $p + q = m$.

Notice that under the change $\omega \mapsto \lambda\omega$ we might swap the signature (p, q) of Ω_ω to (q, p) (if $\lambda < 0$ and $p \neq q$), therefore we add the restriction $p \geq q$ to our definition of the signature of Ω_ω. For local results, observe that it is enough to calculate the signature at ξ_0 to ensure the validity of (A3) in a neighbourhood.

The following result is fundamental in our work; it is analogous to Sylvester's law of inertia [Syl52] adapted to our control-affine systems context. It can also be seen as a by-product of Morse lemma [Mor34] with parameters (see [Hör07, p. 502] for a proof).

Proposition 3.1. *Assume that assumptions (A1), (A2), and (A3) hold. Then, there locally exists a feedback $\beta \in C^\infty(\mathcal{M}, GL_m(\mathbb{R}))$ such that Ω_ω is transformed into,*

$$\tilde{\Omega}_\omega = I_{p,q} = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}. \tag{3.2}$$

Proof. We denote $\Omega_0 = \Omega_\omega|_{\xi_0}$. By Sylvester's law of inertia [Syl52], there exists a matrix $\beta_0 \in GL_m(\mathbb{R})$ such that

$$\beta_0^t \Omega_0 \beta_0 = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}.$$

In order to prevent the eigenvalues to collapse in a neighbourhood we additionally apply $\beta_1 = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & m \end{pmatrix}$. Now we take any smooth extension $\beta(\xi)$ such that $\beta(\xi_0) = \beta_0\beta_1$. Let $\bar{g} = g\beta$ and set $\bar{\Omega}_\omega = (\omega([\bar{g}_i, \mathrm{ad}_f \bar{g}_j]))$, i.e. Ω_ω evaluated on the frame $\bar{g}$. For this frame we have

$$\bar{\Omega}_\omega\big|_{\xi_0} = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 4 & & \\ & & \ddots & \\ & & & m^2 \end{pmatrix}.$$

Hence for a neighbourhood of ξ_0, the map $\bar{\Omega}_\omega$ has distinct eigenvalues and thus can be smoothly diagonalised by an orthogonal transformation (recall that $\bar{\Omega}_\omega$ is symmetric), see [Iva89]. In this frame, we have $\tilde{\Omega}_\omega$ given by

$$\begin{pmatrix} \lambda_1(\xi) & 0 & \\ 0 & \ddots & \\ & & \lambda_m(\xi) \end{pmatrix}$$

and since all our transformation preserved the signature (p, q) of Ω_ω, we may assume $\lambda_i(\xi) > 0$ for $i = 1, \ldots, p$ and $\lambda_i(\xi) < 0$ for $i = p + 1, \ldots, m$. Applying the final feedback $\tilde{g}_i = \frac{1}{\sqrt{\lambda_i}} g_i$ for $i = 1, \ldots, p$ and $\tilde{g}_i = \frac{1}{\sqrt{-\lambda_i}} g_i$ for $i = p + 1, \ldots, m$ gives the desired form. ∎

Notice that the construction of $\tilde{\Omega}_\omega$ of the form (3.2) is purely algebraic, hence in what follows there will be no loss of generality in assuming that the vector fields

$(g_1, \ldots, g_m)$ of a control-affine system Σ (satisfying (A1), (A2), and (A3)) give $\tilde{\Omega}_\omega$ of the form (3.2). For a control-affine system Σ, having the signature of Ω_ω equal to (p, q) is, of course, a necessary condition for its equivalence to $\Sigma_{p,q}$. This condition, however, is not sufficient as we will show in the next chapters.

For our results, we will need to characterise feedback transformations $\tilde{g} = g\beta$ that preserve the form (3.2) of Proposition 3.1. We will deduce that $\beta(\cdot) \in GO(p, q)$, the conformal group, i.e. β satisfies $\beta^t \mathtt{I}_{p,q} \beta = \lambda \mathtt{I}_{p,q}$ for some $\lambda(\cdot) \neq 0$ called the associated multiplier. The following lemmata will be useful for our characterisation and classification results.

Lemma 3.3. *If $\beta \in C^\infty(\mathcal{M}, GO(p, q))$ with associated multiplier λ, then for any vector field $v \in V^\infty(\mathcal{M})$ we have*

$$\mathrm{L}_v(\beta)\,\beta^{-1} - \frac{1}{2\lambda}\mathrm{L}_v(\lambda)\,\mathrm{Id}_m \in \mathit{Lie}\,(O(p, q))\,. \tag{3.3}$$

Proof. If $\beta(\cdot) \in GO(p, q)$, then by definition we have $\beta^t \mathtt{I}_{p,q} \beta = \lambda \mathtt{I}_{p,q}$. By differentiating along v the last relation we obtain

$$\begin{aligned}
\mathrm{L}_v(\beta)^t\,\mathtt{I}_{p,q}\beta + \beta^t\mathtt{I}_{p,q}\mathrm{L}_v(\beta) &= \mathrm{L}_v(\lambda)\,\mathtt{I}_{p,q},\\
\left(\mathrm{L}_v(\beta)\,\beta^{-1}\right)^t\mathtt{I}_{p,q} + \mathtt{I}_{p,q}\left(\mathrm{L}_v(\beta)\,\beta^{-1}\right) &= \mathrm{L}_v(\lambda)\,\beta^{-t}\mathtt{I}_{p,q}\beta^{-1},\\
\left(\mathrm{L}_v(\beta)\,\beta^{-1}\right)^t\mathtt{I}_{p,q} + \mathtt{I}_{p,q}\left(\mathrm{L}_v(\beta)\,\beta^{-1}\right) &= \tfrac{1}{\lambda}\mathrm{L}_v(\lambda)\,\mathtt{I}_{p,q},\\
\left(\mathrm{L}_v(\beta)\,\beta^{-1} - \tfrac{1}{2\lambda}\mathrm{L}_v(\lambda)\,\mathrm{Id}_m\right)^t\mathtt{I}_{p,q} &+ \mathtt{I}_{p,q}\left(\mathrm{L}_v(\beta)\,\beta^{-1} - \tfrac{1}{2\lambda}\mathrm{L}_v(\lambda)\,\mathrm{Id}_m\right) = 0.
\end{aligned}$$

The last equation characterises the elements of $\mathrm{Lie}\,(O(p, q))$. ■

For the following two lemmata, we will work on a smooth n-dimensional manifold $\mathcal{M}$ equipped with local coordinates ξ. The lemma below gives necessary and sufficient conditions for a nonhomogeneous system of quasilinear PDEs to possess smooth local solution.

Lemma 3.4. *Let $\mathcal{G} = \mathrm{span}\,\{g_1, \ldots, g_m\} \subset T\mathcal{M}$ be an involutive distribution of constant rank m. Consider the following system of m quasilinear PDEs*

$$\mathrm{L}_{g_i}(z^r) = B_i^r, \quad 1 \le i \le m, \quad 1 \le r \le N, \tag{3.4}$$

where $B_i^r = B_i^r(\xi, z)$ depend on the unknowns z^r. For any smooth manifold $\mathcal{X}$ such that $T_{\xi_0}\mathcal{X} \oplus \mathcal{G}(\xi_0) = T_{\xi_0}\mathcal{M}$ and any smooth function $\phi : \mathcal{X} \to \mathbb{R}^N$, system (3.4) admits a unique local, around ξ_0, smooth solution $z(\xi)$ such that $z_{|\mathcal{X}} = \phi$ if and only if for all $1 \le i < j \le m$ it holds

$$\mathrm{L}_{g_i}\left(B_j^r\right) - \mathrm{L}_{g_j}\left(B_i^r\right) + \sum_{s=1}^{N} B_i^s \frac{\partial B_j^r}{\partial z^s} - B_j^s \frac{\partial B_i^r}{\partial z^s} = \sum_{k=1}^{m} \nu_{i,j}^k B_k^r, \quad 1 \le r \le N, \tag{3.5}$$

where the functions $\nu_{i,j}^k$ are defined by $[g_i, g_j] = \sum_{k=1}^m \nu_{i,j}^k g_k$.

In the case, where $g_i = \frac{\partial}{\partial \xi^i}$, for $1 \le i \le m$, integrability conditions (3.5) take the form of those of [Fla63, Section 7.4].

Proof. We will show that system (3.4) can be transformed to a classical system of homogeneous linear PDEs to which the classical Frobenius theorem can be applied. Let g_i^k denote the components of the vector fields g_i in coordinates $\xi = (\xi^1, \dots, \xi^n)$, that is $g_i = \sum_{k=1}^n g_i^k \frac{\partial}{\partial \xi^k}$. In the extended space (ξ, z), we look for solutions in the implicit form $\psi(\xi, z) = 0 \in \mathbb{R}^N$, with $\operatorname{rk} \frac{\partial \psi}{\partial z} = N$. We differentiate the equation $\psi(\xi, z(\xi)) = 0$ along the fields g_i:

$$
\begin{aligned}
\sum_{k=1}^n g_i^k \frac{\partial}{\partial \xi^k} (\psi) &= \sum_{k=1}^n g_i^k \frac{\partial \psi}{\partial \xi^k} + g_i^k \sum_s \frac{\partial \psi}{\partial z^s} \frac{\partial z^s}{\partial \xi^k} = 0 \\
&= \sum_{k=1}^n g_i^k \frac{\partial \psi}{\partial \xi^k} + \sum_s \frac{\partial \psi}{\partial z^s} B_i^s = 0 \\
&= \mathrm{L}_{\mathsf{G}_i} (\psi) = 0,
\end{aligned}
$$

where $\mathsf{G}_i = g_i + \sum_s B_i^s \frac{\partial}{\partial z^s}$. Thus, we get a homogeneous system of PDEs for ψ which, by the Frobenius theorem, possesses solutions, for any fixed $\phi : \mathcal{X} \to \mathbb{R}^N$, if and only if the distribution $G = \operatorname{span}\{\mathsf{G}_1, \dots, \mathsf{G}_m\}$ is involutive (equivelently, possesses integral manifolds of maximal dimension m). Hence, we compute (using summation over s)

$$
\begin{aligned}
[\mathsf{G}_i, \mathsf{G}_j] &= \left[g_i + B_i^s \frac{\partial}{\partial z^s}, g_j + B_j^s \frac{\partial}{\partial z^s} \right] \\
&= [g_i, g_j] + \mathrm{L}_{g_i} \left(B_j^s\right) \frac{\partial}{\partial z^s} - \mathrm{L}_{g_j} \left(B_i^s\right) \frac{\partial}{\partial z^s} + \left[B_i^s \frac{\partial}{\partial z^s}, B_j^s \frac{\partial}{\partial z^s} \right] \\
&= \sum_{k=1}^m \nu_{i,j}^k g_k + \left(\mathrm{L}_{g_i} \left(B_j^s\right) - \mathrm{L}_{g_j} \left(B_i^s\right) \right) \frac{\partial}{\partial z^s} + \left[B_i^s \frac{\partial}{\partial z^s}, B_j^s \frac{\partial}{\partial z^s} \right]
\end{aligned}
$$

and thus G is involutive if and only if (using again summation over s)

$$
\left(\mathrm{L}_{g_i} \left(B_j^s\right) - \mathrm{L}_{g_j} \left(B_i^s\right) \right) \frac{\partial}{\partial z^s} + \left[B_i^s \frac{\partial}{\partial z^s}, B_j^s \frac{\partial}{\partial z^s} \right] = \sum_{k=1}^m \nu_{i,j}^k B_k^s \frac{\partial}{\partial z^s},
$$

equivalently, for all i, j, r it holds

$$
\mathrm{L}_{g_i} \left(B_j^r\right) - \mathrm{L}_{g_j} \left(B_i^r\right) + \sum_s B_i^s \frac{\partial B_j^r}{\partial z^s} - B_j^s \frac{\partial B_i^r}{\partial z^s} = \sum_{k=1}^m \nu_{i,j}^k B_k^r.
$$

∎

The next lemma relates the pseudo-orthogonal group and its Lie algebra through solutions of a system of nonhomogeneous linear partial differential equations.

Lemma 3.5. *Let $\mathcal{G} = \operatorname{span}\{g_1, \dots, g_m\} \subset T\mathcal{M}$ be an involutive distribution of constant rank m, let $\mathcal{X}$ be a smooth submanifold such that $T_{\xi_0}\mathcal{X} \oplus \mathcal{G}(\xi_0) = T_{\xi_0}\mathcal{M}$. Suppose that $\beta \in C^\infty(\mathcal{M}, GL_m(\mathbb{R}))$ satisfy*

$$
\mathrm{L}_{g_i} (\beta) = -\mu_i \beta, \quad \forall\, 1 \le i \le m, \tag{3.6}
$$

with $\mu_i(\cdot) \in Lie(O(p,q))$, and $\beta_{|\mathcal{X}} = \beta_0 \in C^\infty(\mathcal{X}, O(p,q))$. Then, locally around ξ_0, we have $\beta \in C^\infty(\mathcal{M}, O(p,q))$.

Proof. First, observe that the complementarity between $\mathcal{G}$ and $T\mathcal{X}$ implies that the solution of (3.6) is unique (see the above lemma). Second, notice that

$$\mathrm{L}_{g_i}(\beta)\,\beta^{-1} + \beta\mathrm{L}_{g_i}\left(\beta^{-1}\right) = 0 \Rightarrow \mathrm{L}_{g_i}\left(\beta^{-1}\right) = -\beta^{-1}\mathrm{L}_{g_i}(\beta)\,\beta^{-1},$$

and third, the relation $\mu_i^t \mathtt{I}_{p,q} + \mathtt{I}_{p,q}\mu_i = 0$ yields $\mathtt{I}_{p,q}\mu_i^t = -\mu_i \mathtt{I}_{p,q}$. Next, we set $\Theta = \mathtt{I}_{p,q}\beta^{-t}\mathtt{I}_{p,q}$, which satisfies

$$\begin{aligned}\mathrm{L}_{g_i}(\Theta) &= \mathtt{I}_{p,q}\mathrm{L}_{g_i}\left(\beta^{-1}\right)^t \mathtt{I}_{p,q} = -\mathtt{I}_{p,q}\beta^{-t}\mathrm{L}_{g_i}(\beta)^t\,\beta^{-t}\mathtt{I}_{p,q}\\ &= \mathtt{I}_{p,q}\mu_i^t\beta^{-t}\mathtt{I}_{p,q} = -\mu_i\mathtt{I}_{p,q}\beta^{-t}\mathtt{I}_{p,q} = -\mu_i\Theta.\end{aligned}$$

Moreover, observe that $\Theta_{|\mathcal{X}} = \mathtt{I}_{p,q}\beta_0^{-t}\mathtt{I}_{p,q} = \beta_0$. Therefore, β and Θ satisfy the same system of equations with the same initial condition implying $\beta = \Theta$. So, we deduce $\beta\mathtt{I}_{p,q}\beta^t = \mathtt{I}_{p,q}$, i.e. $\beta(\cdot) \in O(p,q)$. ■

By a direct calculation, we give as a corollary of Lemma 3.4 the integrability conditions for system (3.6).

Corollary 3.1. *Under the previous assumptions denote* $z^r = z^{r_1,r_2} = \beta^{r_1}_{r_2}$ *and* $B_i^r = B_i^{r_1,r_2} = -\left(\mu_i\beta\right)^{r_1}_{r_2}$*, where* μ_i *is a smooth* $m \times m$*-matrix, and* $1 \le r \le N = m^2$*. Then, system (3.6) admits smooth solution if and only if*

$$\mathrm{L}_{g_i}(\mu_j) - \mathrm{L}_{g_j}(\mu_i) + \mu_i\mu_j - \mu_j\mu_i = \sum_{k=1}^{m} \nu_{i,j}^k \mu_k. \tag{3.7}$$

Observe that the integrability condition (3.7) can be recovered by computing

$$\mathrm{L}_{[g_i,g_j]}(\beta) = \mathrm{L}_{g_i}\left(\mathrm{L}_{g_j}(\beta)\right) - \mathrm{L}_{g_j}\left(\mathrm{L}_{g_i}(\beta)\right),$$

which, indeed, gives (3.7) because the right hand side $-\mu_i\beta$ of (3.6) is linear with respect to the unknown β.

Chapter 4

Control systems with paraboloid quadric nonholonomic constraints

In this chapter, we extend some of the results of Chapter 2 to the case of a smooth manifold $\mathcal{X}$ of dimension $\dim \mathcal{X} = 3$. The results of this chapter will be generalised to an arbitrary dimension in the next chapter and we present them in the present chapter in a detailed way for two reasons. First, they will serve as an introduction to the general concepts developed in the preceding chapter and the reader could familiarise with them. And, second, the low dimension of the problems allows us to obtain more precise specifications about the structure of the considered problems.

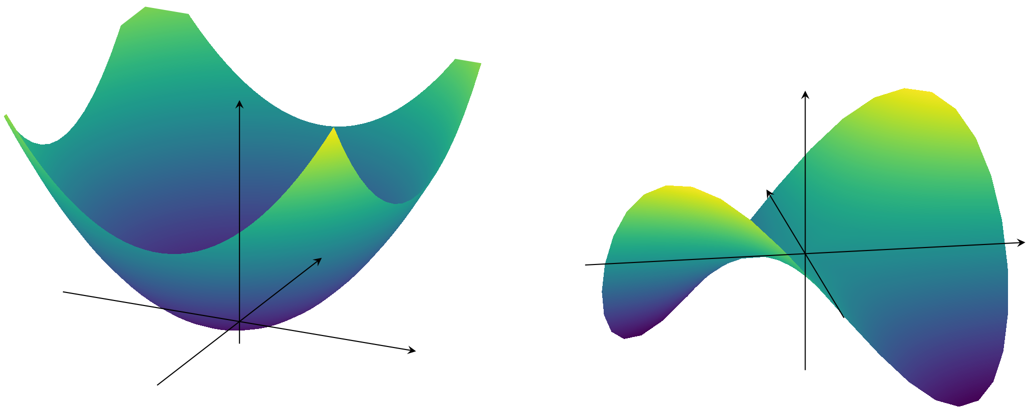
In the tangent bundle $T\mathcal{X}$ of a 3-dimensional manifold $\mathcal{X}$ we consider a smooth submanifold

$$\mathcal{S} = \{(x, \dot{x}) \in T\mathcal{X},\ S(x, \dot{x}) = 0\}$$

satisfying $\operatorname{rk} \frac{\partial S}{\partial \dot{x}}(x, \dot{x}) = 1$ for all $(x, \dot{x}) \in \mathcal{S}$. We equip $\mathcal{X}$ with local coordinates $x = (z, y)$, with $\dim(z) = 1$ and $\dim(y) = 2$. Our first purpose is to characterise the equivalence (in the sense of Definition 1.5 of Chapter 1) of $\mathcal{S}$ to a (p, q)-parabolic submanifold given by

$$\mathcal{S}_Q = \left\{ \dot{z} = \dot{y}^t Q(x) \dot{y} + 2b(x)\dot{y} + c(x) \right\},$$

where $Q(x)$ is a smooth 2 by 2 symmetric matrix of full rank with signature (p, q), $b(x) = (b_1(x), b_2(x))$ is a smooth covector, and $c(x)$ is a smooth function. The matrix $Q(x)$ can be seen as a degenerate metric g on $\mathcal{X}$ such that $\ker \mathsf{g} = \operatorname{span}\left\{\frac{\partial}{\partial z}\right\}$. We represent the submanifold $\mathcal{S}_Q$ by the 4-tuple $\left(\frac{\partial}{\partial z}, Q, b, c\right)$. Throughout the chapter, we assume that the signature of Q is constant in a neighbourhood(it is a consequence of Q being full-rank). Clearly the sign of the determinant of Q identifies its signature. Up to the transformation $\tilde{z} = -z$, there are only two classes of (p, q)-parabolic submanifolds. First, we conseider $\operatorname{sgn}(Q) = (2, 0)$ and call $\mathcal{S}_Q$ a p-elliptic submanifold, and, second, we have $\operatorname{sgn}(Q) = (1, 1)$ and we call $\mathcal{S}_Q$ a p-hyperbolic submanifold. In our nomenclature, the «p-» stands for *paraboloid* in accordance with the terminology of quadrics in affine geometry. We will denote $\mathcal{S}_{pE}$, resp. $\mathcal{S}_{pH}$, a p-elliptic, resp. a p-hyperbolic, submanifold; figure 4.1 shows the graph of a $\mathcal{S}_{pE}$ and of a $\mathcal{S}_{pH}$ in a fiber $T_{x_0}\mathcal{X}$.



(a) Elliptic paraboloid (b) Hyperbolic paraboloid

Figure 4.1: Illustration of the two classes paraboloid submanifolds in $T_{x_0}\mathcal{X}$

Recall that to any submanifold $\mathcal{S}$ we can attach two parametrisations: $\Xi_{\mathcal{S}}$ (first prolongation) and $\Sigma_{\mathcal{S}}$ (second prolongation) given by

$$\Xi_{\mathcal{S}} \,:\, \dot{x} = F(x,w), \quad \text{and} \quad \Sigma_{\mathcal{S}} \,:\, \begin{cases} \dot{x} &= F(x,w) \\ \dot{w} &= u \end{cases},$$

where $F(x,w)$ is a smooth map satisfying $S(x,F(x,w))$ for all w. Those prolongations are seen as control-nonlinear systems and control-affine systems, respectively. From Proposition 1.6 of Chapter 1, we know that the equivalence of submanifolds under diffeomorphism and multiplication by a nonvanishing function is equivalent to the equivalence of the corresponding first and second prolongation under feedback transformations. Thus, the problem of characterising submanifolds $\mathcal{S}_Q$ is dealt with under the characterisation of their second prolongation defined by

$$\Sigma_{\mathcal{S}_Q} \,:\, \begin{cases} \dot{z} &= w^t Q(x) w + b(x) w + c(x) \\ \dot{y} &= w \\ \dot{w} &= u \end{cases}$$

and we will denote $\Sigma_{\mathcal{S}_{pE}}$, resp. $\Sigma_{\mathcal{S}_{pH}}$, a second prolongation of $\mathcal{S}_{pE}$, resp. $\mathcal{S}_{pH}$. Since $\dim \mathcal{X} = 3$ and $\operatorname{rk} \frac{\partial S}{\partial \dot{x}}(x,\dot{x}) = 1$ this second prolongation lives of a 5-dimensional manifold $\mathcal{M} \cong \mathcal{X} \times \mathbb{R}^2$ and is parametrised by 2 controls $u = (u_1, u_2)$. The following definition gives an alternative description of second prolongations of p-ellitpic and p-hyperbolic submanifolds.

Definition 4.1 ((p,q)-parabolisable systems)**.** We say that a control-affine system Σ on a 5-dimensional manifold $\mathcal{M}$ and with 2 controls, is (p,q)-*parabolisable* if it is feedback equivalent to $\Sigma_Q^{p,q}$ given by

$$\Sigma_{p,q} \,:\, \begin{cases} \dot{x} &= A(x)\left((w_1)^2 \pm (w_2)^2\right) + B(x)w + C(x) \\ \dot{w} &= u \end{cases} \qquad (x,w) \in \mathcal{M},\ u \in \mathbb{R}^2,$$

where «+» in $\pm$ corresponds to $(p,q) = (2,0)$ and «−» to $(p,q) = (1,1)$, A, $B = (B_1, B_2)$, and C are smooth vector fields on $\mathcal{M}/_{\sim}$, where the equivalence relation is defined by the integrable distribution $\operatorname{span}\left\{\frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}\right\}$, satisfying $A \wedge B_1 \wedge B_2 \neq 0$.

Lemma 3.1 of Chapter 3 shows that $\Sigma_{2,0}$, resp. $\Sigma_{1,1}$, is feedback equivalent to $\Sigma_{\mathcal{S}_{pE}}$, resp. $\Sigma_{\mathcal{S}_{pH}}$. This justifies the following terminology: we denote $\Sigma_{2,0}$ by Σ_{pE} and call it a *p-elliptic* system and we denote $\Sigma_{1,1}$ by Σ_{pH} and call it a *p-hyperbolic* system. and we will replace the problem of characterising $\mathcal{S}_Q$ by that of Σ_{pE} and of Σ_{pH}.

In the previous chapter, we introduce some general conditions that are necessary for the equivalence of an arbitrary control-affine system Σ to Σ_{pE} or Σ_{pH}. We briefly recall them here. Consider a control affine system $\Sigma = (f, g)$ with state space $\mathcal{M}$ a 5-dimensional manifold and with 2 controls. We attached to Σ the following two distributions

$$\mathcal{D}^0 = \operatorname{span}\{g_1, g_2\}, \quad \text{and} \quad \mathcal{D}^1 = \operatorname{span}\{g_1, g_2, \operatorname{ad}_f g_1, \operatorname{ad}_f g_2\}.$$

The first two necessary assumptions for the equivalence of Σ to Σ_{pE} or Σ_{pH} are

(A1) The distribution $\mathcal{D}^0$ is involutive and has constant rank 2,

(A2) The distribution $\mathcal{D}^1$ has constant rank 4.

Under those assumptions, we construct a bilinear map $\Omega_\omega : \mathcal{D}^0 \times \mathcal{D}^0 \to \mathbb{R}$ and we assume

(A3) $\operatorname{sgn}(\Omega_\omega) = (p, q)$ is constant and satisfies $p + q = 2$.

Proposition 3.1 of Chapter 3 shows that we can always choose (with algebraic transformations only) a suitable frame of $\mathcal{D}^0$ such that Ω_ω takes locally the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \pm 1 \end{smallmatrix}\right)$.

In the following sections, we will fully characterise the systems Σ_{pE} and Σ_{pH} via algebraic and differential relations of structure functions attached to control-affine systems. After this, we will treat the problem of classification of p-elliptic and p-hyperbolic systems and provide several normal and canonical forms.

1 Study of p-Elliptic systems

In this section, we fully characterise p-elliptic systems represented by the normal form

$$\Sigma_{pE} : \begin{cases} \dot{x} &= A(x)\left(w_1^2 + w_2^2\right) + B_1(x)w_1 + B_2(x)w_2 + C(x) \\ \dot{w} &= u \end{cases},$$

where A, B_1, B_2, and C are smooth vector fields on the 3-dimensional manifold $\mathcal{M}/\mathcal{D}^0$. Recall from the previous section that systems Σ_{pE} are second prolongations of p-elliptic submanifolds $\mathcal{S}_{pE}$ and that a characterisation of Σ_{pE} induces a characterisation of $\mathcal{S}_{pE}$. In this section, whenever we make a reference to assumption (A3) we always mean

$(\mathrm{A3})_{\mathrm{pE}}$ $\operatorname{sgn}(\Omega_\omega) = (2, 0)$.

In the following subsections, we will, first, give a complete characterisation of p-elliptic systems in terms of checkable algebraic and differential relations of structure functions attached to control-affine systems and, second, working within the class of p-elliptic systems we will give a classification, under the action of feedback transformations, of p-elliptic systems.

1.1 Characterisation of p-elliptic systems

Consider a control-affine system $\Sigma : \dot{\xi} = f(\xi) + g_1(\xi)u_1 + g_2(\xi)u_2$, with state $\xi \in \mathcal{M}$, a smooth 5-dimensional manifold. For the system $\Sigma = (f, g)$ we define the following distributions

$$\mathcal{D}^0 = \operatorname{span}\{g_1, g_2\}, \quad \text{and} \quad \mathcal{D}^1 = \operatorname{span}\{g_1, g_2, \mathrm{ad}_f g_1, \mathrm{ad}_f g_2\}.$$

Recall that assumptions (A1) and (A2) imply that $\mathcal{D}^0$ is involutive and of constant rank 2 and that $\mathcal{D}^1$ is of constant rank 4. Moreover, $(A3)_{\mathrm{pE}}$ implies that $\mathcal{D}^1$ is not involutive, otherwise, Ω_ω would identically vanish.

Definition 4.2 (Weak orthonormal frame)**.** We say that the pair (g_1, g_2) is a *weak orthonormal frame* of $\mathcal{D}^0 = \operatorname{span}\{g_1, g_2\}$, shortly WOF, if

$$[g_1, \mathrm{ad}_f g_1] - [g_2, \mathrm{ad}_f g_2] = 0 \mod \mathcal{D}^1, \quad \text{and} \quad [g_1, \mathrm{ad}_f g_2] = [g_2, \mathrm{ad}_f g_1] = 0 \mod \mathcal{D}^1.$$

The terminology *orthonormal* is justified by the fact that if (g_1, g_2) is a WOF, then for any ω the matrix Ω_ω, with respect to (g_1, g_2), is the matrix of a homothety, that is given by $\Omega_\omega = \lambda \cdot \mathrm{Id}$, where $\lambda > 0$ (since $\operatorname{sgn}(\Omega_\omega) = (2, 0)$) is a smooth function that depends on ω. Therefore, if (g_1, g_2) is a WOF, then the fields (g_1, g_2) have the same «length», i.e. $\Omega_\omega(g_1, g_1) = \Omega_\omega(g_2, g_2)$, and are «orthogonal» $\Omega_\omega(g_1, g_2) = 0$. Assume that (g_1, g_2) is a WOF, then we can uniquely define structure functions μ^k and $\mu^k_{i,j}$, for $i, j, k \in \{1, 2\}$, by

$$\begin{aligned} [g_1, \mathrm{ad}_f g_1] - [g_2, \mathrm{ad}_f g_2] &= \mu^1 \mathrm{ad}_f g_1 + \mu^2 \mathrm{ad}_f g_2 \mod \mathcal{D}^0, \\ [g_i, \mathrm{ad}_f g_j] &= \mu^1_{i,j} \mathrm{ad}_f g_1 + \mu^2_{i,j} \mathrm{ad}_f g_2 \mod \mathcal{D}^0, \quad i \neq j. \end{aligned}$$

The structure functions $\mu^k_{i,j}$ can be related to the structure function (ν^1, ν^2), defined by

$$[g_1, g_2] = \nu^1 g_1 + \nu^2 g_2,$$

by the Jacobi identity applied to $[g_1, \mathrm{ad}_f g_2]$, namely,

$$\nu^1 = \mu^1_{1,2} - \mu^1_{2,1}, \quad \text{and} \quad \nu^2 = \mu^2_{1,2} - \mu^2_{2,1}. \tag{4.1}$$

Proposition 4.1 (Existence and properties of weak orthonormal frames)**.**

(i) Under assumptions (A1), (A2), and $(A3)_{pE}$ there exists a weak orthonormal frame.

(ii) If (g_1, g_2) is a WOF, then $(\tilde{g}_1, \tilde{g}_2) = (g_1, g_2)\beta$ is also a WOF if and only if β is of the form

$$\beta(\lambda, \theta) = \lambda \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \tag{4.2}$$

up to a permutation of the fields (g_1, g_2), and where $\lambda = \lambda(\xi) > 0$ and $\theta = \theta(\xi)$ are smooth functions.

(iii) *Under the feedback $(g_1, g_2) = (\tilde{g}_1, \tilde{g}_2)\beta(\lambda, \theta)$, with β of the form (4.2), the structure functions μ^k and $\mu^k_{i,j}$ of (g_1, g_2) and the structure functions $\tilde{\mu}^k$ and $\tilde{\mu}^k_{i,j}$ of $(\tilde{g}_1, \tilde{g}_2)$ are related by*

$$
\begin{aligned}
(4.3) \quad \begin{pmatrix} \mu^1 & \mu^2 \\ \mu^1_{1,2} & \mu^2_{1,2} \\ \mu^1_{2,1} & \mu^2_{2,1} \end{pmatrix} &= \begin{pmatrix} \frac{1}{\lambda}\mathrm{L}_{g_1}(\lambda) + \mathrm{L}_{g_2}(\theta) & \mathrm{L}_{g_1}(\theta) - \frac{1}{\lambda}\mathrm{L}_{g_2}(\lambda) \\ -\mathrm{L}_{g_1}(\theta) & \frac{1}{\lambda}\mathrm{L}_{g_1}(\lambda) \\ \frac{1}{\lambda}\mathrm{L}_{g_2}(\lambda) & \mathrm{L}_{g_2}(\theta) \end{pmatrix} \\
&+ \begin{pmatrix} \cos(2\theta) & \sin(2\theta) & \sin(2\theta) \\ -\frac{1}{2}\sin(2\theta) & \cos(\theta)^2 & -\sin(\theta)^2 \\ -\frac{1}{2}\sin(2\theta) & -\sin(\theta)^2 & \cos(\theta)^2 \end{pmatrix} \begin{pmatrix} \tilde{\mu}^1 & \tilde{\mu}^2 \\ \tilde{\mu}^1_{1,2} & \tilde{\mu}^2_{1,2} \\ \tilde{\mu}^1_{2,1} & \tilde{\mu}^2_{2,1} \end{pmatrix} \beta(\lambda, \theta).
\end{aligned}
$$

Proof.

(i) Consider a control-affine system Σ given by vector fields f and $g = (g_1, g_2)$ and suppose that assumptions (A1), (A2), and $(\mathrm{A3})_{\mathrm{pE}}$ hold. By Proposition 3.1, there exists a feedback β such that for the new frame $\tilde{g} = g\beta$ we have $\tilde{\Omega}_\omega = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Clearly $\tilde{g}$ is a weak orthonormal frame.

(ii) Assume that (g_1, g_2) and $(\tilde{g}_1, \tilde{g}_2) = (g_1, g_2)\beta$ are two weak orthonormal frames, for them we have $\Omega_\omega = \eta\,\mathrm{Id}$ and $\tilde{\Omega}_\omega = \tilde{\eta}\,\mathrm{Id}$, respectively, with $\eta > 0$ and $\tilde{\eta} > 0$. Using relation (3.1) we see that β must satisfy

$$\tilde{\eta}\,\mathrm{Id} = \eta\,\beta^t \mathrm{Id}\beta = \eta\beta^t\beta.$$

Hence, β is a similitude, that is $\beta \in C^\infty(\mathcal{M}, GO(2, \mathbb{R}))$. Since every element of the group of similitude is the direct product of an homothety and of an isometry we, finally, have

$$\text{either} \quad \beta_1 = \lambda \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \quad \text{or} \quad \beta_2 = \lambda \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix},$$

where $\lambda = \pm\sqrt{\frac{\tilde{\eta}}{\eta}}$. In both cases we have $\beta_i(\lambda, \theta + \pi) = \beta_i(-\lambda, \theta)$ thus we can consider $\lambda > 0$ only. Finally we see that $\beta_2(\lambda, \theta) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \beta_1\left(\lambda, \frac{\pi}{2} - \theta\right)$, that is β_2 is of the same type as β_1 up to a permutation of the fields (g_1, g_2). Thus it will be convenient to restrict the transformations to feedback β of the form β_1 only.

(iii) The computation is detailed in Appendix 4.A.

■

We denote by $C^+(2)$ the group of admissible feedback transformations $C^\infty(\mathcal{M}, \mathbb{R}^{*+}) \times C^\infty(\mathcal{M}, SO(2, \mathbb{R}))$ acting on the set of weak orthonormal frames. Every $\beta \in C^+(2)$ is denoted by $\beta(\lambda, \theta)$ to emphasis the two types of transformations available, and observe that we have $\beta^{-1} = \beta\left(\frac{1}{\lambda}, -\theta\right)$. Under basic assumptions (A1), (A2), and $(\mathrm{A3})_{\mathrm{pE}}$ there always exists a weak orthonormal frame and we are now going to reinforce this notion which will turn out to be the key of the characterisation of p-elliptic systems.

Definition 4.3 (Strong orthonormal frame)**.** We say that a pair (g_1, g_2) is a *strong orthonormal frame* of $\mathcal{D}^0 = \operatorname{span}\{g_1, g_2\}$, shortly SOF, if

$$[g_1, \mathrm{ad}_f g_1] - [g_2, \mathrm{ad}_f g_2] = 0 \mod \mathcal{D}^0, \quad \text{and} \quad [g_1, \mathrm{ad}_f g_2] = [g_2, \mathrm{ad}_f g_1] = 0 \mod \mathcal{D}^0.$$

Clearly, this definition implies a more rigid structure than a weak orthonormal frame as it requires the orthonormality properties to hold modulo a distribution of rank two. In other words, a strong orthonormal frame is a weak orthonormal frame with structure functions satisfying $\mu^k = \mu^k_{i,j} = 0$.

Proposition 4.2 (Properties of the strong orthonormal frames)**.**

(i) *Any p-elliptic system Σ_{pE} possesses a strong orthonormal frame,*

(ii) *If $(\tilde{g}_1, \tilde{g}_2)$ is a strong orthonormal frame, then $(g_1, g_2) = (\tilde{g}_1, \tilde{g}_2)\beta(\lambda, \theta)$ is a weak orthonormal frame whose structure functions $\mu^k_{i,j}$ and μ^k satisfy*

$$\begin{pmatrix} \mu^1 & \mu^2 \\ \mu^1_{1,2} & \mu^2_{1,2} \\ \mu^1_{2,1} & \mu^2_{2,1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda}\mathrm{L}_{g_1}(\lambda) + \mathrm{L}_{g_2}(\theta) & \mathrm{L}_{g_1}(\theta) - \frac{1}{\lambda}\mathrm{L}_{g_2}(\lambda) \\ -\mathrm{L}_{g_1}(\theta) & \frac{1}{\lambda}\mathrm{L}_{g_1}(\lambda) \\ \frac{1}{\lambda}\mathrm{L}_{g_2}(\lambda) & \mathrm{L}_{g_2}(\theta) \end{pmatrix}. \tag{4.4}$$

(iii) *If (g_1, g_2) is a strong orthonormal frame, then g_1 and g_2 are commuting vector fields,*

(iv) *If (g_1, g_2) is a strong orthonormal frame then $(\tilde{g}_1, \tilde{g}_2) = (g_1, g_2)\beta$ is also a strong orthonormal frame if and only if $\beta \in C^+(2)$ and $\beta(\lambda, \theta)$, additionally, satisfies*

$$\mathrm{L}_{\tilde{g}_1}(\lambda) = \mathrm{L}_{\tilde{g}_2}(\lambda) = \mathrm{L}_{\tilde{g}_1}(\theta) = \mathrm{L}_{\tilde{g}_2}(\theta) = 0. \tag{4.5}$$

Proof.

(i) Recall that Σ_{pE} is given by the vector fields $(g_1, g_2) = \left(\frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}\right)$ and $f = A(x)(w_1^2 + w_2^2) + B(x)w + C(x) \mod \mathcal{D}^0$. Then it is a straightforward computation to show that (g_1, g_2) is a strong orthonormal frame for Σ_{pE}:

$$[g_1, \mathrm{ad}_f g_1] - [g_2, \mathrm{ad}_f g_2] = 2A(x) - 2A(x) = 0, \quad \text{and} \quad [g_1, \mathrm{ad}_f g_2] = [g_2, \mathrm{ad}_f g_1] = 0.$$

(ii) Assume that $(\tilde{g}_1, \tilde{g}_2)$ is a strong orthonormal frame and let $(g_1, g_2) = (\tilde{g}_1, \tilde{g}_2)\beta(\lambda, \theta)$, then the result follows from the application of relation (4.3) with $\tilde{\mu}^k_{i,j} = \tilde{\mu}^k = 0$.

(iii) Recall that the structure functions (ν^1, ν^2) are defined by $[g_1, g_2] = \nu^1 g_1 + \nu^2 g_2$. Assume that (g_1, g_2) is a strong orthonormal frame, then by (4.1) we have $\nu^1 = \nu^2 = 0$, thus $[g_1, g_2] = 0$.

(iv) Assume (g_1, g_2) and $(\tilde{g}_1, \tilde{g}_2)$ are strong orthonormal frames. In particular they are both weak orthonormal frames, therefore they must differ by a feedback $\beta \in C^+(2)$ and by relation (4.4) with $\tilde{\mu}^k_{i,j} = \tilde{\mu}^k = 0$ we see that the functions λ and θ of the feedback necessary satisfy relation (4.5).

■

We have now set everything for a characterisation, in terms of algebraic and differential relations between the structure functions, of p-elliptic systems. Statement *(i)* of the above proposition asserts that Σ_{pE} possess a strong orthonormal frame thus the existence of a SOF is a necessary condition for the equivalence of Σ to Σ_{pE}. Therefore, the structure functions $\mu^k_{i,j}$ and μ^k of a weak orthonormal frame (g_1, g_2) which is equivalent via $\beta(\lambda, \theta)$ to a SOF have to satisfy some necessary relations. From (4.4) we first have

$$\mu^2_{1,2} + \mu^2_{2,1} \; = \mu^1 \qquad \text{and} \quad \mu^1_{1,2} + \mu^1_{2,1} = -\mu^2. \tag{4.6}$$

Moreover those structure function are related to the feedback transformation $\beta(\lambda, \theta)$ by the following systems of first order partial differential equations.

$$\begin{cases} \mathrm{L}_{g_1}(\theta) & = -\mu^1_{1,2} \\ \mathrm{L}_{g_2}(\theta) & = \mu^2_{2,1} \end{cases}, \qquad \begin{cases} \frac{1}{\lambda}\mathrm{L}_{g_1}(\lambda) & = \mu^2_{1,2} \\ \frac{1}{\lambda}\mathrm{L}_{g_2}(\lambda) & = \mu^1_{2,1} \end{cases}. \tag{4.7}$$

In particular, the structure functions have to fulfil the following integrability conditions of the systems (4.7):

$$\begin{aligned} \mathrm{L}_{g_1}\left(\mu^2_{2,1}\right) + \mathrm{L}_{g_2}\left(\mu^1_{1,2}\right) & = -\nu^1\mu^1_{1,2} + \nu^2\mu^2_{2,1}, \\ \mathrm{L}_{g_1}\left(\mu^1_{2,1}\right) - \mathrm{L}_{g_2}\left(\mu^2_{1,2}\right) & = \nu^1\mu^2_{1,2} + \nu^2\mu^1_{2,1}, \end{aligned} \tag{4.8}$$

respectively, for θ and $\ln\lambda$ (recall that $\lambda > 0$ in our feedback transformations), where ν^1 and ν^2 are defined by $[g_1, g_2] = \nu^1 g_1 + \nu^2 g_2$. Observe that relations (4.6) and (4.8) are algebraic and differential and thus can explicitly be tested on any given weak orthonormal frame.

The following theorem shows, first that those conditions are also sufficient for the existence of a SOF and, second, that the existence of a SOF fully characterises p-elliptic systems Σ_{pE}.

Theorem 4.4 (Characterisation of p-elliptic systems)**.** *Consider a control-affine system Σ satisfying assumptions (A1), (A2), and (A3)$_{pE}$. Then, the following statements are locally equivalent,*

(pE1) *Σ is feedback equivalent to Σ_{pE};*

(pE2) *For any weak orthonormal frame (g_1, g_2) of Σ, the structure functions $\mu^k_{i,j}$ and μ^k satisfy (4.6) and, moreover, the systems given by (4.7) have solutions;*

(pE3) *For any weak orthonormal frame (g_1, g_2) of Σ, the structure functions $\mu^k_{i,j}$ and μ^k satisfy (4.6) and (4.8);*

(pE4) *There exists a strong orthonormal frame of Σ;*

Proof. We will show (pE1)⇒(pE2)⇒(pE3)⇒(pE4)⇒(pE1).

(pE1) $\Longrightarrow$ (pE2) $\Longrightarrow$ (pE3). This is the analysis performed above the theorem.

(pE3) $\Longrightarrow$ (pE4). Assume that (g_1, g_2) is a weak orthonormal frame with structure functions $\mu^k_{i,j}$ and μ^k satisfying (4.6) and (4.8). Given a solution (λ, θ) of (4.7), construct the feedback

$$\beta\left(\frac{1}{\lambda}, -\theta\right) = \frac{1}{\lambda}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

notice that we can always have $\lambda > 0$ by solving the second system of (4.7) in terms of $\ln(\lambda)$. Then, the computation in Appendix 4.B shows that the new fields $(\tilde{g}_1, \tilde{g}_2) = (g_1, g_2)\beta\left(\frac{1}{\lambda}, -\theta\right)$ form a strong orthonormal frame.

(pE4) $\implies$ (pE1). Let (g_1, g_2) be a strong orthonormal frame, recall that by statement *(iii)* of Proposition 4.2 this frame satisfies $[g_1, g_2] = 0$. We introduce coordinates $(x, w) = \phi(\xi)$ such that $\phi_\star g_i = \frac{\partial}{\partial w_i}$, for $i = 1, 2$. After applying a suitable feedback $f \mapsto f + \alpha_1 g_1 + \alpha_2 g_2$ the system Σ takes the form

$$\begin{cases} \dot{x} & = \mathsf{f}(x, w) \\ \dot{w}_1 & = u_1 \\ \dot{w}_2 & = u_2 \end{cases},$$

for which $(g_1, g_2) = \left(\frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}\right)$ is a strong orthonormal frame. By definition of a strong orthonormal frame, we have the following conditions on $\mathsf{f}(x, w)$:

$$\frac{\partial^2 \mathsf{f}}{\partial w_1 \partial w_2} = 0 \quad \text{and} \quad \frac{\partial^2 \mathsf{f}}{\partial w_1^2} - \frac{\partial^2 \mathsf{f}}{\partial w_2^2} = 0.$$

Solutions of those equations admit the following closed form,

$$\mathsf{f} = A(x)(w_1^2 + w_2^2) + B_1(x)w_1 + B_2(x)w_2 + C(x)$$

where the A, B_1, B_2, and C are smooth vector fields on $\mathcal{M}/\mathcal{D}^0$. For this form we have $\mathcal{D}^1 = \operatorname{span}\{2Aw_1 + B_1, 2Aw_2 + B_2\} \mod \mathcal{D}^0$ which, by assumption (A2), is of constant rank 4, and since we have preserved the signature of Ω_ω in all our operations we conclude that $A \notin \mathcal{D}^1$ and thus we have $A \wedge B_1 \wedge B_2 \neq 0$. ∎

The above characterisation of p-elliptic systems requires two conditions. Condition (4.8) is differential and asserts that there exists a feedback (λ, θ) that transforms a WOF into a new frame (g_1, g_2) that is *orthogonal* with respect to $\mathcal{D}^0$. Then, condition (4.8) ensures that this frame is, actually, also *orthonormal*.

Existence of a strong orthonormal frame of $\mathcal{D}^0$ is a characterisation of p-elliptic systems. Statement *(iv)* of Proposition 4.2 imply that two such frame differ by a feedback $\beta(\lambda, \theta)$ that is constant on the leaves of the distribution $\mathcal{D}^0$ (giving another justification for statement *(i)* of Proposition 4.3 below). In the following subsection, we will work in the class of p-elliptic system and give a classification including normal and canonical forms of those systems.

1.2 Classification of p-elliptic systems

We now investigate the problem of classification of p-elliptic submanifolds $S_{pE} \subset T\mathcal{X}$. This problem is replaced by the equivalent problem of classification of their first prolongations defined by

$$\Xi_{pE} \,:\, \dot{x} = A(x)(w_1^2 + w_2^2) + B_1(x)w_1 + B_2(x)w_2 + C(x),$$

where $w = (w_1, w_2)^t$ plays the role of a control that enters in a nonlinear way, and A, B_1, B_2, and C are smooth vector fields on $\mathcal{X}$. We are interested in systems satisfying $A \wedge B_1 \wedge B_2 \neq 0$. A p-elliptic system Ξ_{pE} is represented by the 4-tuple (A, B_1, B_2, C) of vector fields. We will describe several orbits of the action of feedback transformations given by $\tilde{x} = \phi(x)$ and $w = \psi(x, \tilde{w})$. First, we have the following characterisation of admissible feedback transformations.

Proposition 4.3 (Equivalence of p-elliptic systems)**.**

(i) *If two p-elliptic systems* $\Xi_{pE} = (A, B_1, B_2, C)$ *and* $\tilde{\Xi}_{pE} = (\tilde{A}, \tilde{B}_1, \tilde{B}_2, C)$ *are feedback equivalent via a diffeomorphism* $\tilde{x} = \phi(x)$ *and an invertible feedback transformation* $w = \psi(x, \tilde{w})$*, then* $\psi = \alpha(x) + \beta(\lambda, \theta)\tilde{w}$ *where* $\alpha(x) = (\alpha_1(x), \alpha_2(x))^t$ *and* $\beta(\lambda, \theta)$ *is a smooth matrix of the form*

$$\beta(\lambda, \theta) = \lambda \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},$$

where $\lambda = \lambda(x)$ *and* $\theta = \theta(x)$ *are smooth functions satisfying* $\lambda > 0$*. Moreover,*

$$\begin{aligned} \tilde{A} &= \phi_* \left(\lambda^2 A\right), \\ \tilde{B}_1 &= \phi_* \left(2\lambda \left(\alpha_1 \cos(\theta) + \alpha_2 \sin(\theta)\right) A + \lambda \cos(\theta) B_1 + \lambda \sin(\theta) B_2\right), \\ \tilde{B}_2 &= \phi_* \left(2\lambda \left(-\alpha_1 \sin(\theta) + \alpha_2 \cos(\theta)\right) A - \lambda \sin(\theta) B_1 + \lambda \cos(\theta) B_2\right), \\ \tilde{C} &= \phi_* \left(C + \left((\alpha_1)^2 + (\alpha_2)^2\right) A + \alpha_1 B_1 + \alpha_2 B_2\right). \end{aligned} \tag{4.9}$$

(ii) *Conversely, if a diffeomorphism* $\tilde{x} = \phi(x)$ *and a 4-tuple of functions* $(\alpha_1, \alpha_2, \lambda, \theta)$ *on* $\mathcal{X}$*, with* $\lambda > 0$*, satisfy (4.9), then the feedback transformation* $\tilde{x} = \phi(x)$ *and* $\psi(x, \tilde{w}) = \alpha(x) + \beta(\lambda, \theta)\tilde{w}$ *brings* Ξ_{pE} *into* $\tilde{\Xi}_{pE}$*.*

Remark (Locality of the results). When we introduced the definition of p-elliptic systems Ξ_{pE}, we assumed that this form holds locally around an arbitrary point (x_0, w_0). We see, in statement *(i)* of the above proposition, that the pure feedback transformations $w = \psi(x, \tilde{w})$ that conjugate p-elliptic systems are global with respect to w. Therefore, in all results below, we will consider the form Ξ_{pE} *locally* around x_0 and *globally* in w. ◆

Proof.

(i) Clearly diffeomorphisms of $\mathcal{X}$ map p-elliptic systems into p-elliptic systems and we have to show that the pure feedback transformations $w = \psi(x, \tilde{w})$ that conjugate p-elliptic systems are of the form $w = \alpha(x) + \beta(\lambda, \theta)\tilde{w})$. To this end, we apply $(w_1, w_2) = (\psi_1(x, \tilde{w}), \psi_2(x, \tilde{w}))$ to Ξ_{pE}, which yields

$$\dot{x} = A(\psi_1^2 + \psi_2^2) + B_1\psi_1 + B_2\psi_2 + C. \tag{4.10}$$

Since the vector fields $A = A(x)$ and $B_i = B_i(x)$ are linearly independent, we conclude that in order to preserve the p-elliptic structure of the system, the functions $\psi_1^2 + \psi_2^2$, ψ_1, and ψ_2 have to satisfy the following conditions:

(a) $\frac{\partial^3 \psi_i}{\partial \tilde{w}_j^3} = \frac{\partial^2 \psi_i}{\partial \tilde{w}_1 \partial \tilde{w}_2} = 0$, for $i, j = 1, 2$, i.e. ψ_i has to be a polynomial of degree at most 2 in $\tilde{w}$ and without mixed term $\tilde{w}_1\tilde{w}_2$,

(b) $\frac{\partial^3 (\psi_1^2+\psi_2^2)}{\partial \tilde{w}_j^3} = \frac{\partial^2 (\psi_1^2+\psi_2^2)}{\partial \tilde{w}_1 \partial \tilde{w}_2} = 0$, for $j = 1, 2$, i.e. the same as above for the function $\psi_1^2 + \psi_2^2$,

(c) $\frac{\partial^2 (\psi_1^2+\psi_2^2)}{\partial \tilde{w}_1^2} = \frac{\partial^2 (\psi_1^2+\psi_2^2)}{\partial \tilde{w}_2^2}$, i.e. we keep the same coefficient before the quadratic expression, Solutions of those conditions are given by $\psi = \alpha + \beta\tilde{w}$ with $\alpha = (\alpha_1(x), \alpha_2(x))^t$ and β of the form

$$\text{either} \quad \beta = \begin{pmatrix} \beta_1 & -\beta_2 \\ \beta_2 & \beta_1 \end{pmatrix}, \quad \text{or} \quad \beta = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & -\beta_1 \end{pmatrix}.$$

Observe that the latter form of β is the same as the former up to permutation of the role of w_1 and w_2, thus we shall restrict feedback transformations to the former form of β only. Since feedback transformations are invertible, we have $\det(\beta) = (\beta_1)^2 + (\beta_2)^2 \neq 0$. Thus, we set $\lambda = (\beta_1)^2 + (\beta_2)^2$ and θ as a solution of $\cos(\theta) = \frac{\beta_1}{\lambda}$ and $\sin(\theta) = \frac{\beta_2}{\lambda}$ to obtain the required form of feedback transformations. Secondly, establishing relation (4.9) is a straightforward computation from (4.10) using $\psi = \alpha + \beta(\lambda, \theta)\tilde{w}$ and identifying quadratic and affine terms.

(ii) Conversely, for ϕ and $(\alpha_1, \alpha_2, \lambda, \theta)$ satisfying (4.9), we clearly establish feedback equivalence of Ξ_{pE} and $\tilde{\Xi}_{pE}$ via $\tilde{x} = \phi(x)$ and $w = \alpha + \beta(\lambda, \theta)\tilde{w}$.

■

We will develop relations involving structure functions attached uniquely to the tuple (A, B_1, B_2, C) only and thus independent of the action of diffeomorphisms of $\mathcal{X}$. So we will act on $\Xi_{pE} = (A, B_1, B_2, C)$ by $(\alpha, \lambda, \theta)$ and we will denote $(\tilde{A}, \tilde{B}_1, \tilde{B}_2, \tilde{C})$ the result of that action (given by (4.9) with $\phi = \mathrm{Id}$) and call it a *reparametrisation*. For a p-elliptic system Ξ_{pE}, we call the triple (A, B_1, B_2) a p-elliptic frame (pE-frame, shortly) and we introduce the structure functions μ_i^k and ν^k for $i = 1, 2$ and $k = 0, 1, 2$ defined by the following brackets expressions

$$[A, B_i] = \mu_i^0 A + \mu_i^1 B_1 + \mu_i^2 B_2, \quad [B_1, B_2] = \nu^0 A + \nu^1 B_1 + \nu^2 B_2.$$

Definition 4.5 (Types of p-elliptic frames)**.** For pE-frames (A, B_1, B_2), we define the following subclasses

(a) *pseudo-commutative pE-frames* if $[A, B_i] = 0 \mod \operatorname{span}\{A\}$, that is $\mu_i^1 = \mu_i^2 = 0$ for $i = 1, 2$;

(b) *almost-commutative pE-frames* if $[A, B_i] = [B_1, B_2] = 0 \mod \operatorname{span}\{A\}$, that is $\mu_i^1 = \mu_i^2 = \nu^1 = \nu^2 = 0$ for $i = 1, 2$;

(c) *commutative pE-frames* if $[A, B_i] = [B_1, B_2] = 0$, that is $\mu_i^k = \nu^k = 0$ for $i = 1, 2$ and $k = 0, 1, 2$.

Clearly each class is a subclass of the next one and observe that by (4.9) the distribution $\operatorname{span}\{A\}$ is uniquely attached to a p-elliptic system, thus pseudo and almost-commutative pE-frames are well-defined. Moreover, reparametrisations act on C by adding a linear combination of (A, B_1, B_2), it suggests to introduce the following decomposition

$$C = \gamma^0 A + \gamma^1 B_1 + \gamma^2 B_2,$$

with structure functions γ^k, for $k = 0, 1, 2$. The following technical lemma shows how the three sets of structure functions μ_i^k, ν^k, and γ^k are transformed under reparametrisations $(\alpha, \lambda, \theta)$. Recall that $\lambda > 0$ and it will be convenient to denote $\Lambda := \ln(\lambda)$.

Lemma 4.1 (Structure functions transformations). *Let $\Xi_{pE} = (A, B_1, B_2, C)$ and $\tilde{\Xi}_{pE} = (\tilde{A}, \tilde{B}_1, \tilde{B}_2, \tilde{C})$ be two feedback equivalent p-elliptic systems with structure functions μ_i^k, ν^k, γ^k, and $\tilde{\mu}_i^k$, $\tilde{\nu}^k$, $\tilde{\gamma}^k$, respectively. Then, we have the following formulae*

$$
\begin{aligned}
\tilde{\mu}_1^0 &= -2\mathrm{L}_{\tilde{B}_1}(\Lambda) + \lambda\cos(\theta)\left(\mu_1^0 - 2\alpha_1\mu_1^1 - 2\alpha_2\mu_1^2 + 2\mathrm{L}_A(\alpha_1)\right)\\
&\qquad +\lambda\sin(\theta)\left(\mu_2^0 - 2\alpha_1\mu_2^1 - 2\alpha_2\mu_2^2 + 2\mathrm{L}_A(\alpha_2)\right),\\
\tilde{\mu}_2^0 &= -2\mathrm{L}_{\tilde{B}_2}(\Lambda) - \lambda\sin(\theta)\left(\mu_1^0 - 2\alpha_1\mu_1^1 - 2\alpha_2\mu_1^2 + 2\mathrm{L}_A(\alpha_1)\right)\\
&\qquad +\lambda\cos(\theta)\left(\mu_2^0 - 2\alpha_1\mu_2^1 - 2\alpha_2\mu_2^2 + 2\mathrm{L}_A(\alpha_2)\right),\\
\tilde{\mu}_1^1 &= \lambda^2\left(\mathrm{L}_A(\Lambda) + \cos^2(\theta)\mu_1^1 + \sin^2(\theta)\mu_2^2 + \cos(\theta)\sin(\theta)(\mu_1^2 + \mu_2^1)\right),\\
\tilde{\mu}_2^2 &= \lambda^2\left(\mathrm{L}_A(\Lambda) + \cos^2(\theta)\mu_2^2 + \sin^2(\theta)\mu_1^1 - \cos(\theta)\sin(\theta)(\mu_1^2 + \mu_2^1)\right),\\
\tilde{\mu}_1^2 &= \lambda^2\left(\mathrm{L}_A(\theta) + \cos^2(\theta)\mu_1^2 - \sin^2(\theta)\mu_2^1 - \cos(\theta)\sin(\theta)(\mu_1^1 - \mu_2^2)\right),\\
\tilde{\mu}_2^1 &= \lambda^2\left(-\mathrm{L}_A(\theta) + \cos^2(\theta)\mu_2^1 - \sin^2(\theta)\mu_1^2 - \cos(\theta)\sin(\theta)(\mu_1^1 - \mu_2^2)\right),
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
\tilde{\nu}^0 &= \nu^0 - 2\alpha_1\nu^1 - 2\alpha_2\nu^2 + 2\alpha_1\mu_2^0 - 2\alpha_2\mu_1^0 + 4\alpha_1\alpha_2(\mu_1^1 - \mu_2^2)\\
&\quad +4(\alpha_2)^2\mu_1^2 - 4(\alpha_1)^2\mu_2^1 + 4\left(\alpha_1\mathrm{L}_A(\alpha_2) - \alpha_2\mathrm{L}_A(\alpha_1)\right)\\
&\quad +2\mathrm{L}_{B_1}(\alpha_2) - 2\mathrm{L}_{B_2}(\alpha_1),\\
\tilde{\nu}^1 &= \lambda\cos(\theta)\left(\nu^1 - 2\alpha_2\mu_1^1 + 2\alpha_1\mu_2^1 - 2\alpha_2\mathrm{L}_A(\Lambda) - 2\alpha_1\mathrm{L}_A(\theta)\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. -\mathrm{L}_{B_1}(\theta) - \mathrm{L}_{B_2}(\Lambda)\right)\\
&\quad +\lambda\sin(\theta)\left(\nu^2 - 2\alpha_2\mu_1^2 + 2\alpha_1\mu_2^2 + 2\alpha_1\mathrm{L}_A(\Lambda) - 2\alpha_2\mathrm{L}_A(\theta)\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. -\mathrm{L}_{B_2}(\theta) + \mathrm{L}_{B_1}(\Lambda)\right),\\
\tilde{\nu}^2 &= -\lambda\sin(\theta)\left(\nu^1 - 2\alpha_2\mu_1^1 + 2\alpha_1\mu_2^1 - 2\alpha_2\mathrm{L}_A(\Lambda) - 2\alpha_1\mathrm{L}_A(\theta)\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. -\mathrm{L}_{B_1}(\theta) - \mathrm{L}_{B_2}(\Lambda)\right)\\
&\quad +\lambda\cos(\theta)\left(\nu^2 - 2\alpha_2\mu_1^2 + 2\alpha_1\mu_2^2 + 2\alpha_1\mathrm{L}_A(\Lambda) - 2\alpha_2\mathrm{L}_A(\theta)\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. -\mathrm{L}_{B_2}(\theta) + \mathrm{L}_{B_1}(\Lambda)\right),
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
\tilde{\gamma}^0 &= \tfrac{1}{\lambda^2}\left(\gamma^0 - (\alpha_1)^2 - (\alpha_2)^2 - 2\alpha_1\gamma_1 - 2\alpha_2\gamma^2\right),\\
\tilde{\gamma}^1 &= \tfrac{1}{\lambda}\left(\cos(\theta)\left(\gamma^1 + \alpha_1\right) + \sin(\theta)\left(\gamma^2 + \alpha_2\right)\right),\\
\tilde{\gamma}^2 &= \tfrac{1}{\lambda}\left(-\sin(\theta)\left(\gamma^1 + \alpha_1\right) + \cos(\theta)\left(\gamma^2 + \alpha_2\right)\right).
\end{aligned}
\tag{4.13}
$$

Moreover, the following relations between the structure functions always hold:

$$
\begin{aligned}
\mathrm{L}_A(\nu^0) - \mathrm{L}_{B_1}(\mu_2^0) + \mathrm{L}_{B_2}(\mu_1^0) &= \nu^0(\mu_1^1 + \mu_2^2) - \nu^1\mu_1^0 - \nu^2\mu_2^0,\\
\mathrm{L}_A(\nu^1) - \mathrm{L}_{B_1}(\mu_2^1) + \mathrm{L}_{B_2}(\mu_1^1) &= \nu^1\mu_2^2 - \nu^2\mu_2^1 - \mu_2^0\mu_1^1 + \mu_1^0\mu_2^1,\\
\mathrm{L}_A(\nu^2) - \mathrm{L}_{B_1}(\mu_2^2) + \mathrm{L}_{B_2}(\mu_1^2) &= -\nu^1\mu_1^2 + \nu^2\mu_1^1 - \mu_2^0\mu_1^2 + \mu_1^0\mu_2^2.
\end{aligned}
\tag{4.14}
$$

Proof. The computations are quite long and tedious so we leave them for Appendix 4.C. ∎

In the following proposition we are going to characterise pE-systems of the form

$$
\Xi_{pE}^d \;:\; \dot{x} = \begin{pmatrix}1\\0\\0\end{pmatrix}\left((w_1)^2 + (w_2)^2\right) + \begin{pmatrix}0\\q_1\\0\end{pmatrix} w_1 + \begin{pmatrix}0\\0\\r_2\end{pmatrix} w_2 + C,
$$

where $q_1 = q_1(x)$ and $r_2 = r_2(x)$. This pre-normal form is interesting as it describes a first prolongation of p-elliptic submanifolds S_{pE} for which the matrix Q is diagonalised (see Corollary 4.2 for details).

Proposition 4.4 (Characterisation of Ξ_{pE}^d). *For a p-elliptic system $\Xi_{pE} = (A, B_1, B_2, C)$ the following statements are equivalent:*

(i) Ξ_{pE} is locally feedback equivalent to Ξ_{pE}^d;

(ii) There exists a reparametrisation $(\alpha, \lambda, \theta)$ *such that*

$$\left[\tilde{A}, \tilde{B}_1\right] = 0 \mod \operatorname{span}\left\{\tilde{A}, \tilde{B}_1\right\}, \quad \text{and} \quad \left[\tilde{A}, \tilde{B}_2\right] = 0 \mod \operatorname{span}\left\{\tilde{A}, \tilde{B}_2\right\};$$

(iii) The structure functions μ_i^k *of the pE-frame* (A, B_1, B_2) *satisfy*

$$\mathrm{L}_A\left(\frac{\mu_1^2+\mu_2^1}{\mu_1^1-\mu_2^2}\right) = \left(\mu_2^1-\mu_1^2\right)\left[1+\left(\frac{\mu_1^2+\mu_2^1}{\mu_1^1-\mu_2^2}\right)^2\right]. \tag{4.15}$$

Remark. If in relation (4.15) we have $\mu_1^1-\mu_2^2=0$ but $\mu_1^2+\mu_2^1\neq 0$ then we can simply use the condition

$$\mathrm{L}_A\left(\frac{\mu_1^1-\mu_2^2}{\mu_1^2+\mu_2^1}\right) = -\left(\mu_2^1-\mu_1^2\right)\left[1+\left(\frac{\mu_1^1-\mu_2^2}{\mu_1^2+\mu_2^1}\right)^2\right]. \tag{4.15'}$$

If we have $\mu_1^1-\mu_2^2=0$ and also $\mu_1^2+\mu_2^1=0$, then we fall into the case of the Theorem 4.6 which characterises a special form of Ξ_{pE}^d. ◆

Proof. The proof of the implication *(i)*⇒*(ii)* is immediate by a straightforward calculation on the pE-frame $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$ attached to Ξ_{pE}^d.

(ii)⇒*(iii)*. Assume that (A, B_1, B_2) is equivalent to $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$ via a reparametrisation $(\alpha, \lambda, \theta)$. Then, using relation (4.11) with $\tilde{\mu}_1^2=\tilde{\mu}_2^1=0$ we obtain

$$\begin{cases} \ \ \mathrm{L}_A(\theta)+\cos^2(\theta)\mu_1^2-\sin^2(\theta)\mu_2^1-\cos(\theta)\sin(\theta)(\mu_1^1-\mu_2^2) &= 0 \\ -\mathrm{L}_A(\theta)+\cos^2(\theta)\mu_2^1-\sin^2(\theta)\mu_1^2-\cos(\theta)\sin(\theta)(\mu_1^1-\mu_2^2) &= 0 \end{cases},$$

which yields $\cos(2\theta)(\mu_1^2+\mu_2^1)-\sin(2\theta)(\mu_1^1-\mu_2^2)=0$ and $2\mathrm{L}_A(\theta)+\mu_1^2-\mu_2^1=0$. Hence,

$$\tan(2\theta)=\frac{\mu_1^2+\mu_2^1}{\mu_1^1-\mu_2^2}, \quad \text{and} \quad 2\mathrm{L}_A(\theta)=\mu_2^1-\mu_1^2$$

and, finally, by differentiating the first relations along A we obtain relation (4.15).

(iii)⇒*(ii)*. Choose θ satisfying $\tan(2\theta)=\frac{\mu_1^2+\mu_2^1}{\mu_1^1-\mu_2^2}$, i.e. $\sin(2\theta)(\mu_1^1-\mu_2^2)-\cos(2\theta)(\mu_1^2+\mu_2^1)=0$, and differentiating this relation along A and using condition (4.15), we obtain that $\mathrm{L}_A(\theta)=\frac{1}{2}(\mu_2^1-\mu_1^2)$. Applying the reparametrisation $\alpha=0$, $\lambda=1$, and θ as above, we obtain, using (4.11), that the new pE-frame $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$ satisfies $\tilde{\mu}_1^2=\tilde{\mu}_2^1=0$.

(ii)⇒*(i)*. Assume that the pE-frame $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$ of Ξ_{pE} satisfies $\left[\tilde{A}, \tilde{B}_1\right]=0$ mod span$\left\{\tilde{A}, \tilde{B}_1\right\}$ and $\left[\tilde{A}, \tilde{B}_2\right]=0$ mod span$\left\{\tilde{A}, \tilde{B}_2\right\}$. Introduce coordinates $(z, \tilde{y}_1, \tilde{y}_2)=\phi(x)$ such that $\phi_*\tilde{A}=\frac{\partial}{\partial z}$,

$$B_1=\phi_*\tilde{B}_1=\begin{pmatrix} q_0 \\ q_1 \\ q_2 \end{pmatrix}, \quad \text{and} \quad B_2=\phi_*\tilde{B}_2=\begin{pmatrix} r_0 \\ r_1 \\ r_2 \end{pmatrix}.$$

By assumption, we have $\rho\,[A, B_1]=B_1 \mod \operatorname{span}\{A\}$ and $\varrho\,[A, B_2]=B_2 \mod \operatorname{span}\{A\}$, with $\rho=\rho(x)$ and $\varrho=\varrho(x)$. Thus we have $q_i=\rho\frac{\partial q_i}{\partial z}$ and $r_i=\varrho\frac{\partial r_i}{\partial z}$, for $i=1,2$, yielding $q_i = Q_i(\tilde{y})\exp(D(x))$ and $r_i = R_i(\tilde{y})\exp(G(x))$. Since $A \wedge B_1 \wedge B_2 \neq 0$ we deduce that the distributions $\mathcal{Q} = \mathrm{span}\left\{Q_1(\tilde{y})\frac{\partial}{\partial \tilde{y}_1} + Q_2(\tilde{y})\frac{\partial}{\partial \tilde{y}_2}\right\}$ and $\mathcal{R} = \mathrm{span}\left\{R_1(\tilde{y})\frac{\partial}{\partial \tilde{y}_1} + R_2(\tilde{y})\frac{\partial}{\partial \tilde{y}_2}\right\}$, living on the $\tilde{y}$-space, can simultaneously be rectified, i.e. there exist coordinates (y_1, y_2) such that $\mathrm{span}\{\mathrm{d}y_1\} = \mathrm{ann}(\mathcal{R})$ and $\mathrm{span}\{\mathrm{d}y_2\} = \mathrm{ann}(\mathcal{Q})$ (see Corollary A.1 of Appendix A). With respect to this new coordinate system, Ξ_{pE} takes the form

$$\dot{x} = \begin{pmatrix}1\\0\\0\end{pmatrix}((w_1)^2 + (w_2)^2) + \begin{pmatrix}q_0\\q_1\\0\end{pmatrix} w_1 + \begin{pmatrix}r_0\\0\\r_2\end{pmatrix} w_2 + C,$$

and applying the transformation $(\tilde{w}_1, \tilde{w}_2) = \left(w_1 + \frac{q_0}{2}, w_2 + \frac{r_0}{2}\right)$, we obtain Ξ^d_{pE}. ∎

We now, state our main classifications results. We start with a characterisation, via relations between the structure functions, of the following normal form

$$\Xi'_{pE} \,:\, \dot{x} = \begin{pmatrix}1\\0\\0\end{pmatrix}((w_1)^2 + (w_2)^2) + \begin{pmatrix}0\\1\\0\end{pmatrix} w_1 + \begin{pmatrix}0\\0\\1\end{pmatrix} w_2 + C,$$

which clearly corresponds to the existence of a commutative pE-frame.

Theorem 4.6 (Existence of a commutative pE-frame)**.** *Consider a p-elliptic system Ξ_{pE} with pE-frame (A, B_1, B_2) and with structure functions μ_i^k, for $i = 1, 2$ and $k = 0, 1, 2$. Then, the following statements are equivalents:*

- *(i) Ξ_{pE} is feedback equivalent to Ξ'_{pE};*
- *(ii) There exists a feedback reparametrisation $(\alpha, \lambda, \theta)$ such that $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$ is a commutative pE-frame;*
- *(iii) There exists a feedback reparametrisation $(\alpha, \lambda, \theta)$ such that $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$ is an almost-commutative pE-frame;*
- *(iv) There exists a feedback reparametrisation $(\alpha, \lambda, \theta)$ such that $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$ is a pseudo-commutative pE-frame;*
- *(v) The structure functions μ_i^k satisfy*

$$\mu_1^1 - \mu_2^2 = 0, \quad \textit{and} \quad \mu_1^2 + \mu_2^1 = 0. \tag{4.16}$$

Observe that a general pE-frame defines nine structure function μ_i^k, ν^k and that the existence of a commutative pE-frame require that only two of them are determined by others (relation (4.16) above). That is, we have to normalise seven structure functions using four feedback functions, namely λ, θ, α_1, and α_2, and respecting the three Jacobi relations (4.14).

Proof. We will show *(i)* ⇒ *(v)* ⇒ *(iv)* ⇒ *(iii)* ⇒ *(ii)* ⇒ *(i)*.
The implication *(i)* ⇒ *(v)* is an immediate consequence of (4.11) with $\tilde{\mu}_1^1 = \tilde{\mu}_2^2 = \tilde{\mu}_1^2 = \tilde{\mu}_2^1 = 0$.

Conversely, for (v) $\Rightarrow$ (iv), consider a pE-frame (A, B_1, B_2) with structure functions μ_i^k and ν^k satisfying (4.16). Take any smooth solution (Λ, θ) of the following independent first order partial differential equations,

$$\mathrm{L}_A(\theta) = -\mu_1^2, \quad \text{and} \quad \mathrm{L}_A(\Lambda) = -\mu_1^1,$$

and apply the reparametrisation $\alpha = 0$ and (λ, θ). Using relation (4.11) we obtain that the new frame $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$ satisfies $\tilde{\mu}_i^j = 0$ for $i, j = 1, 2$, hence it is pseudo-commutative.

(iv) $\Rightarrow$ (iii). Assume that $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$ is a pseudo-commutative frame. Observe that due to the last two relations of (4.14) we have $\mathrm{L}_{\tilde{A}}(\tilde{\nu}^i) = 0$, for $i = 1, 2$. We claim that there exist smooth solutions of the following systems

$$\mathrm{L}_{\tilde{A}}(\Lambda) = 0, \quad \mathrm{L}_{\tilde{A}}(\theta) = 0, \quad \begin{cases} \mathrm{L}_{\tilde{B}_1}(\theta) + \mathrm{L}_{\tilde{B}_2}(\Lambda) & = \tilde{\nu}^1 \\ \mathrm{L}_{\tilde{B}_2}(\theta) - \mathrm{L}_{\tilde{B}_1}(\Lambda) & = \tilde{\nu}^2 \end{cases}.$$

The latter system can be seen as non-homogeneous Cauchy-Riemann equations in the space of leaves of the distribution span$\left\{\tilde{A}\right\}$. Thanks to Jack Lee [Lee20], we have the interpretation that this system translates the fact that some metric is conformally flat (in the sense that the metric is conform to the euclidean metric). The latter problem can always be solved in two dimension, which means that we can bring the system into the classical (that is, with commuting $\tilde{B}_1$ and $\tilde{B}_2$) non-homogeneous Cauchy-Riemann equations, for which smooth solutions always exist (see e.g. [Kra17, theorem 10.1.2]). Applying the reparametrisation (λ, θ) given by any smooth solutions gives a new pE-frame $(\bar{A}, \bar{B}_1, \bar{B}_2)$ satisfying $\bar{\mu}_i^j = \bar{\nu}^j = 0$, for $i, j = 1, 2$; thus that frame is almost-commutative.

(iii) $\Rightarrow$ (ii). Assume that $(\bar{A}, \bar{B}_1, \bar{B}_2)$ is an almost-commutative pE-frame. Applying the reparametrisation $(\alpha, 1, 0)$, where (α_1, α_2) is any smooth solution of the system of equations

$$2\mathrm{L}_A(\alpha_1) = -\bar{\mu}_1^0, \quad \text{and} \quad 2\mathrm{L}_A(\alpha_2) = -\bar{\mu}_2^0,$$

we obtain a new pE-frame $(\hat{A}, \hat{B}_1, \hat{B}_2)$ with $\hat{\mu}_i^k = \hat{\nu}^j = 0$, for $i, j = 1, 2$ and $k = 0, 1, 2$. Finally, observe that the first relation of (4.14) implies $\mathrm{L}_{\hat{A}}(\hat{\nu}^0) = 0$. Now take any solution α_1 of the system (whose integrability condition is indeed, $\mathrm{L}_{\hat{A}}(\hat{\nu}^0) = 0$)

$$\begin{cases} \mathrm{L}_{\hat{A}}(\alpha_1) & = 0 \\ 2\mathrm{L}_{\hat{B}_2}(\alpha_1) & = \hat{\nu}^0, \end{cases}$$

and apply the reparametrisation $(\alpha_1, 0, \lambda = 1, \theta = 0)$ to obtain a pE-frame (A, B_1, B_2) which is commutative.

(ii) $\Rightarrow$ (i). Consider a p-elliptic system Ξ_{pE} such that its pE-frame (A, B_1, B_2) is commutative. Apply a diffeomorphism $(z, y_1, y_2) = \phi(x)$ satisfying $\phi_* A = \frac{\partial}{\partial z}$, $\phi_* B_1 = \frac{\partial}{\partial y_1}$, and $\phi_* B_2 = \frac{\partial}{\partial y_2}$. In those coordinates Ξ_{pE} takes the form Ξ'_{pE}. ■

Remark (Summary of the construction of a commutative pE-frame). Under relation (4.16), the proof (v) $\Rightarrow$ (ii) of the above theorem consists in, successively, constructing a feedback $(\alpha, \lambda, \theta)$ solution of the following systems of first order partial differential equations

$$\left\{\begin{array}{rl} \mathrm{L}_A(\Lambda) & = -\mu_1^1 \\ \mathrm{L}_A(\theta) & = -\mu_1^2 \\ \mathrm{L}_{B_1}(\theta) + \mathrm{L}_{B_2}(\Lambda) & = \nu^1 \\ \mathrm{L}_{B_2}(\theta) - \mathrm{L}_{B_1}(\Lambda) & = \nu^2 \end{array}\right.$$

$$2\mathrm{L}_A(\alpha_2) = 2\mathrm{L}_{B_2}(\Lambda) - \mu_2^0 + 2\alpha_1\mu_1^2$$

$$\left\{\begin{array}{rl} \mathrm{L}_A(\alpha_1) & = 2\mathrm{L}_{B_1}(\Lambda) - \mu_1^0 + 2\alpha_2\mu_1^2 \\ 2\mathrm{L}_{B_2}(\alpha_1) & = \nu^0 - 2\alpha_1\nu^1 - 2\alpha_2\nu^2 + 2\mathrm{L}_{B_1}(\alpha_2) - 4\alpha_2\mathrm{L}_{B_1}(\Lambda) + 4\alpha_1\mathrm{L}_{B_2}(\Lambda) \end{array}\right. .$$

Most of the time, integrability conditions are guaranteed by the Jacobi identity (4.14). However, for the system for λ and θ, the difficulty of showing the existence of solutions lies in the fact that the equations for Λ and θ cannot be separated. The interpretation of that system as the conformal equivalence of some metric allows to show that smooth solutions exist. That interpretation will nicely generalise in higher dimension. ◆

Remark (On the explicit construction of a commutative pE-frame). In the proof of the previous theorem observe that we first use (λ, θ) to pass from a pE-frame to an almost commutative pE-frame, that step requires to solve first order partial differential equations and thus, there is no guarantee that we can explicitly construct λ and θ. However the passage from an almost-commutative pE-frame to a commutative pE-frame can be made explicit (in coordinates in which A is rectified and, simultaneously, B_1 and B_2 are rectified modulo span $\{A\}$) as follows. Consider a p-elliptic system Ξ_{pE} with an almost-commutative frame. Thus, in a suitable coordinate system we have

$$\dot{x} = \begin{pmatrix}1\\0\\0\end{pmatrix}\left((w_1)^2 + (w_2)^2\right) + \begin{pmatrix}b_1(x)\\1\\0\end{pmatrix} w_1 + \begin{pmatrix}b_2(x)\\0\\1\end{pmatrix} w_2 + C(x).$$

Using the reparametrisation $(w_1, w_2) = \left(\tilde{w}_1 - \frac{b_1}{2}, \tilde{w}_2 - \frac{b_2}{2}\right)$ we obtain the form Ξ'_{pE} in the same coordinates. ◆

In the remaining part of this subsection, we will fully characterise the following normal forms of p-elliptic systems (special subclasses of Ξ'_{pE}):

$$\Xi''_{pE} \,:\, \dot{x} = \begin{pmatrix}1\\0\\0\end{pmatrix}\left((w_1)^2 + (w_2)^2\right) + \begin{pmatrix}0\\1\\0\end{pmatrix} w_1 + \begin{pmatrix}0\\0\\1\end{pmatrix} w_2 + \begin{pmatrix}c_0(x)\\0\\0\end{pmatrix},$$

$$\Xi'''_{pE} \,:\, \dot{x} = \begin{pmatrix}1\\0\\0\end{pmatrix}\left((w_1)^2 + (w_2)^2\right) + \begin{pmatrix}0\\1\\0\end{pmatrix} w_1 + \begin{pmatrix}0\\0\\1\end{pmatrix} w_2 + \begin{pmatrix}c_0\\0\\0\end{pmatrix}, \quad c_0 \in \mathbb{R}.$$

Recall that p-elliptic systems Ξ_{pE} are parametrisation of p-elliptic submanifolds S_{pE} given by the 4-tuple $\left(\frac{\partial}{\partial z}, Q, b, c\right)$, with Q a symmetric 2 by 2 matrix of constant signature $(2,0)$, b is a covector, and c is a smooth function. Observe that systems Ξ'_{pE} correspond to a parametrisation of p-elliptic submanifold for which the matrix Q is normalised to Id_2. The normal form Ξ''_{pE} is interesting because it represents a parametrisation of the following p-elliptic submanifold

$$\mathcal{S}''_{pE} = \{\dot{z} = \dot{y}_1^2 + \dot{y}_2^2 + c_0(x)\},$$

that is, a p-elliptic submanifold with normalised matrix Q and, additionally, normalised one-form b. The second form Ξ'''_{pE} characterises the submanifold without functional parameters, namely, those that do not depend on the point $x \in \mathcal{X}$ (corresponding control-systems are called trivialisable in [Ser09]). Moreover, we will show that the latter case can always be brought into a canonical form with $c_0 = 0$ or $c_0 = \pm 1$.

Our conditions will be expressed for Ξ_{pE} in terms of relations between structure functions and, therefore, are checkable on any p-elliptic system. However, those conditions are complicated to interpret and it will be convenient to give, as a corollary, the same conditions for the system Ξ'_{pE}, that is, for a commutative pE-frame. For any p-elliptic system Ξ_{pE}, We define the function $\Gamma = \gamma^0 + (\gamma^1)^2 + (\gamma^2)^2$, which is conjugated by diffeomorphisms and is transformed by $\lambda^2\tilde{\Gamma} = \Gamma$ under reparametrisations $(\alpha, \lambda, \theta)$, as it can be computed from (4.13).

Theorem 4.7 (Classification results of p-elliptic systems)**.** *Consider a p-elliptic system $\Xi_{pE} = (A, B_1, B_2, C)$ with structure function $(\mu_i^k, \nu^k, \gamma^k)$. Then we have*

(i) Ξ_{pE} is equivalent to Ξ'_{pE} if and only if

$$\mu_1^1 - \mu_2^2 = 0, \quad \text{and} \quad \mu_1^2 + \mu_2^1 = 0. \tag{4.16}$$

(ii) Ξ_{pE} is equivalent to Ξ''_{pE} if and only if (4.16) hold and, additionally, we have

$$\begin{aligned} -4\left(\gamma^1 \mathrm{L}_A\left(\gamma^2\right) - \gamma^2 \mathrm{L}_A\left(\gamma^1\right)\right) + 2\mathrm{L}_{B_1}\left(\gamma^2\right) - 2\mathrm{L}_{B_2}\left(\gamma^1\right) = \\ \nu^0 + 2\gamma^1\left(\nu^1 - \mu_2^0\right) + 2\gamma^2\left(\nu^2 + \mu_1^0\right) + 4\left((\gamma^1)^2 + (\gamma^2)^2\right)\mu_1^2, \end{aligned} \tag{4.17}$$

$$\begin{aligned} \mathrm{L}_A^2\left(\gamma^1\right) &= \mathrm{L}_A\left(\tfrac{1}{2}\mu_1^0 + \gamma^2\mu_1^2\right) + \mathrm{L}_{B_1}\left(\mu_1^1\right) + \tfrac{1}{2}\mu_1^0\mu_1^1 \\ &\quad -\mu_1^1\left(-\mathrm{L}_A\left(\gamma^1\right) + \gamma^2\mu_1^2\right) - \mu_1^2\left(-\mathrm{L}_A\left(\gamma^2\right) + \tfrac{1}{2}\mu_2^0 + \gamma^1\mu_2^1\right), \\ \mathrm{L}_A^2\left(\gamma^2\right) &= \mathrm{L}_A\left(\tfrac{1}{2}\mu_2^0 + \gamma^1\mu_2^1\right) + \mathrm{L}_{B_2}\left(\mu_1^1\right) + \tfrac{1}{2}\mu_2^0\mu_1^1 \\ &\quad -\mu_2^1\left(-\mathrm{L}_A\left(\gamma^1\right) + \tfrac{1}{2}\mu_1^0 + \gamma^2\mu_1^2\right) - \mu_2^2\left(-\mathrm{L}_A\left(\gamma^2\right) + \gamma^1\mu_2^1\right), \end{aligned} \tag{4.18}$$

$$\begin{aligned} \mathrm{L}_{B_2}\left(\mathrm{L}_A\left(\gamma^1\right)\right) - \mathrm{L}_{B_1}\left(\mathrm{L}_A\left(\gamma^2\right)\right) &= -\nu^0\mu_1^1 \\ &\quad -\mathrm{L}_{B_1}\left(\tfrac{1}{2}\mu_2^0 + \gamma^1\mu_2^1\right) + \nu^1\left(-\mathrm{L}_A\left(\gamma^1\right) + \tfrac{1}{2}\mu_1^0 + \gamma^2\mu_1^2\right) \\ &\quad +\mathrm{L}_{B_2}\left(\tfrac{1}{2}\mu_1^0 + \gamma^2\mu_1^2\right) + \nu^2\left(-\mathrm{L}_A\left(\gamma^2\right) + \tfrac{1}{2}\mu_2^0 + \gamma^1\mu_2^1\right), \\ \mathrm{L}_{B_1}\left(\mathrm{L}_A\left(\gamma^1\right)\right) + \mathrm{L}_{B_2}\left(\mathrm{L}_A\left(\gamma^2\right)\right) &= \nu^0\mu_1^0 \\ &\quad +\mathrm{L}_{B_1}\left(\nu^2 + \tfrac{1}{2}\mu_1^0 + \gamma^2\mu_1^2\right) - \nu^1\left(\nu^1 + \mathrm{L}_A\left(\gamma^2\right) - \tfrac{1}{2}\mu_2^0 - \gamma^1\mu_2^1\right) \\ &\quad -\mathrm{L}_{B_2}\left(\nu^1 - \tfrac{1}{2}\mu_2^0 - \gamma^1\mu_2^1\right) - \nu^2\left(\nu^2 - \mathrm{L}_A\left(\gamma^1\right) + \tfrac{1}{2}\mu_1^0 + \gamma^2\mu_1^2\right). \end{aligned} \tag{4.19}$$

(iii) Ξ_{pE} is equivalent to Ξ'''_{pE} if and only if (4.16), (4.17), (4.18), (4.19) hold and, additionally, we have

$$\begin{aligned} \mathrm{L}_A\left(\Gamma\right) + 2\Gamma\mu_1^1 &= 0, \\ \mathrm{L}_{B_1}\left(\Gamma\right) + 2\Gamma\mathrm{L}_A\left(\gamma^1\right) - \Gamma\left(\mu_1^0 - 2\gamma^1\mu_1^1 + 2\gamma^2\mu_1^2\right) &= 0, \\ \mathrm{L}_{B_2}\left(\Gamma\right) + 2\Gamma\mathrm{L}_A\left(\gamma^2\right) - \Gamma\left(\mu_2^0 + 2\gamma^1\mu_2^1 - 2\gamma^2\mu_2^2\right) &= 0. \end{aligned} \tag{4.20}$$

Remark (Idea of the theorem). The idea behind statement *(ii)* of the above theorem is the following. For Ξ''_{pE}, with structure functions $(\tilde{\mu}_i^k, \tilde{\nu}^k, \tilde{\gamma}_k)$, we have $\tilde{\mu}_i^k = \tilde{\nu}^k = 0$ (i.e. a commutative pE-frame exits) and $\tilde{\gamma}^1 = \tilde{\gamma}^2 = 0$. Relation (4.13) imposes that we have $\alpha_1 = -\gamma^1$ and $\alpha_2 = -\gamma^2$, thus α is fixed and the group of reparametrisation now depends arbitrarily on (λ, θ) only. Conditions (4.16), (4.17),(4.18), and (4.19), describe then the existence of a reparametrisation (λ, θ), α being fixed by the above constraint, such that a commutative pE-frame exists. The construction of this reparametrisation is given by solutions of two systems of three first order partial differential equations (see the proof below) and thus some integrability conditions are required. Those conditions are given by (4.18) and (4.19), notice that only 2 integrability conditions are required for each system (instead of the three that are expected) because one is always fulfilled by the last two relations of (4.14). And, condition (4.17) ensures that for the constructed pE-frame $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$ we have $\tilde{\nu}^0 = 0$.

The idea behind statement *(iii)* is globally the same, the additional condition (4.20) ensures that the resulting function c_0 of Ξ''_{pH} is constant. Indeed for the system Ξ''_{pH} we have $\Gamma = c_0$ and relation (4.20) implies that $\frac{\partial c_0}{\partial z} = \frac{\partial c_0}{\partial y_1} = \frac{\partial c_0}{\partial y_2} = 0$, i.e. c_0 is constant. ◆

Proof.

(i) It is Theorem 4.6.

(ii) Assume that Ξ_{pE} with structure functions $(\mu_i^k, \nu^k, \gamma^k)$ is equivalent to Ξ''_{pE} with structure functions $\tilde{\mu}_i^k = \tilde{\nu}^k = 0$, $\tilde{\gamma}^1 = \tilde{\gamma}^2 = 0$, and $\tilde{\gamma}^0 = c_0(x)$. The necessity of (4.16) is immediate from Theorem 4.6. Using relation (4.13) with $\tilde{\gamma}^1 = \tilde{\gamma}^2 = 0$ we obtain that $\alpha_1 = -\gamma^1$ and $\alpha_2 = -\gamma^2$. Moreover, from (4.11) and (4.12), with $\tilde{\mu}_i^k = \tilde{\nu}^k = 0$, we deduce, first (with $\tilde{\nu}^0 = 0$) the necessity of (4.17), and second the following systems of first order partial differential equations for $\Lambda = \ln(\lambda)$ and θ:

$$\begin{cases} \mathrm{L}_A(\Lambda) & = -\mu_1^1 \\ \mathrm{L}_{B_1}(\Lambda) & = -\mathrm{L}_A(\gamma^1) + \frac{1}{2}\mu_1^0 + \gamma^2\mu_1^2 \\ \mathrm{L}_{B_2}(\Lambda) & = -\mathrm{L}_A(\gamma^2) + \frac{1}{2}\mu_2^0 + \gamma^1\mu_2^1 \end{cases},$$
$$\text{and} \quad \begin{cases} \mathrm{L}_A(\theta) & = -\mu_1^2 \\ \mathrm{L}_{B_1}(\theta) & = \nu^1 + \mathrm{L}_A(\gamma^1) - \frac{1}{2}\mu_1^0 - \gamma^2\mu_1^2 \\ \mathrm{L}_{B_2}(\theta) & = \nu^2 - \mathrm{L}_A(\gamma^2) + \frac{1}{2}\mu_2^0 + \gamma^1\mu_2^1 \end{cases}.$$

Those two systems imply 6 integrability conditions which are necessary, but two are always provided by the last two equations of (4.14) and the four others are given by (4.18) and (4.19).
Conversely, assume that the structure functions μ_i^k satisfy (4.16), (4.17), (4.18), and (4.19). Then, there exists solutions λ and θ of the above systems (since (4.18), (4.19), together with (4.14), form their integrability conditions), and applying the reparametrisation $\binom{w_1}{w_2} = \binom{-\gamma^1}{-\gamma^2} + \beta(\lambda, \theta)\tilde{w}$ yields the system Ξ''_{pE}.

(iii) Assume Ξ_{pE}, with structure functions $(\mu_i^k, \nu^k, \gamma^k)$, is equivalent to Ξ'''_{pE}, with structure functions $\tilde{\mu}_i^k = \tilde{\nu}^k = 0$, $\tilde{\gamma}^1 = \tilde{\gamma}^2 = 0$, and $\tilde{\gamma}^0 = c_0 \in \mathbb{R}$. The necessity of (4.16), (4.17), (4.18), and (4.19) is clear from the previous item of the proof and we show that (4.20) is necessary as well. For Ξ'''_{pE} we have $\tilde{\Gamma} = c_0 \in \mathbb{R}$, and under reparametrisations $(\alpha, \lambda, \theta)$ we get $\tilde{\Gamma} = \frac{\Gamma}{\lambda^2}$, where $\Gamma = \gamma^0 + (\gamma^1)^2 + (\gamma^2)^2$. Differentiating the last equation along $\tilde{A}$, $\tilde{B}_1$, and $\tilde{B}_2$ leads to relation (4.20).

Conversely, assume that Ξ_{pE} satisfies (4.16), (4.17), (4.18), (4.19), and (4.20). Then, by statement *(ii)*, Ξ_{pE} can be brought into form Ξ''_{pE} for which we have $(A, B_1, B_2) = \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right)$ and $\Gamma = c_0(x)$, thus (4.20) implies $\frac{\partial c_0}{\partial z} = \frac{\partial c_0}{\partial y_1} = \frac{\partial c_0}{\partial y_2} = 0$ and finally $c_0 \in \mathbb{R}$, i.e. we actually have the normal form Ξ'''_{pE}.

∎

As announced, we give the conditions of the previous theorem for a commutative pE-frame in order to provide interpretations of them.

Corollary 4.1 (Classification of Ξ'_{pE})**.** *Consider a p-elliptic system $\Xi'_{pE} = (A, B_1, B_2, C)$ with structure functions $(\mu_i^k, \nu^k, \gamma^k) = (0, 0, \gamma^k)$.*

(i) Ξ'_{pE} is equivalent to Ξ''_{pE} if and only if it satsifies

$$\begin{aligned}
&(4.17') && 2\gamma^1 \mathrm{L}_A\left(\gamma^2\right) - 2\gamma^2 \mathrm{L}_A\left(\gamma^1\right) - \mathrm{L}_{B_1}\left(\gamma^2\right) + \mathrm{L}_{B_2}\left(\gamma^1\right) = 0,\\
&(4.18') && \mathrm{L}_A^2\left(\gamma^1\right) = \mathrm{L}_A^2\left(\gamma^2\right) = 0,\\
&(4.19') && \mathrm{L}_A\left(\mathrm{L}_{B_2}\left(\gamma^1\right) - \mathrm{L}_{B_1}\left(\gamma^2\right)\right) = \mathrm{L}_A\left(\mathrm{L}_{B_1}\left(\gamma^1\right) + \mathrm{L}_{B_2}\left(\gamma^2\right)\right) = 0.
\end{aligned}$$

(ii) Ξ'_{pE} is equivalent to Ξ'''_{pE} if and only if (4.17'), (4.19'), and (4.19') hold and, additionally, we have

$$(4.20') \qquad \mathrm{L}_A(\Gamma) = \mathrm{L}_{B_1}(\Gamma) + 2\Gamma \mathrm{L}_A\left(\gamma^1\right) = \mathrm{L}_{B_2}(\Gamma) + 2\Gamma \mathrm{L}_A\left(\gamma^2\right) = 0.$$

Remark (Interpretation of the conditions). Consider the system Ξ'_{pE} with commutative pE-frame (A, B_1, B_2) and with structure functions $\mu_i^k = \nu^k = 0$ and γ^k. Conditions (4.17'), (4.18'), and (4.19') express the fact that there exists a reparametrisation $(\alpha, \lambda, \theta)$ that, both, preserves the commutativity of the pE-frame and ensures that we obtain $\tilde{\gamma}^1 = \tilde{\gamma}^2 = 0$.

In the rectified frame $(A, B_1, B_2) = \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right)$, the meaning of (4.18') and (4.19') is clear, they imply that

$$(4.21) \qquad \begin{aligned} \gamma^1 &= \gamma_1^1(y)z + \gamma_2^1(y) \\ \gamma^2 &= \gamma_1^2(y)z + \gamma_2^2(y) \end{aligned}, \quad \text{with} \quad \frac{\partial \gamma_1^1}{\partial y_1} = -\frac{\partial \gamma_1^2}{\partial y_2} \quad \text{and} \quad \frac{\partial \gamma_1^1}{\partial y_2} = \frac{\partial \gamma_1^2}{\partial y_1}.$$

That is, the function $\gamma_z(\mathbf{y}) := \frac{\partial}{\partial z}\left(\gamma^1 + \mathbf{i}\gamma^2\right) = \gamma_1^1 + \mathbf{i}\gamma_1^2$ is holomorphic with respect to the complex structure $\mathbf{y} = y_1 + \mathbf{i}y_2$. On the other hand, interpretation of condition (4.17'), which gives an additional relation between the functions γ_j^i, is not so clear yet. To summarise, the systems Ξ'_{pE} that are equivalent to Ξ''_{pE} are parametrised by one arbitrary smooth function of 3 variables, namely $\gamma^0(x)$, and two smooth functions γ^1 and γ^2 of the form (4.21) and satisfying (4.17').

Assume now that Ξ'_{pE} additionally satisfies (4.20'). The first condition implies that $\Gamma = \Gamma(y)$, and rewriting the last two equations as $\frac{\partial \ln(\Gamma)}{\partial \mathbf{y}} = -\bar{\gamma}_z$ and $\frac{\partial \ln(\Gamma)}{\partial \bar{\mathbf{y}}} = -\gamma_z$ (replacing Γ by $-\Gamma$, if necessary), we deduce that $\ln(\Gamma)$ is harmonic, i.e. $\Delta \ln(\Gamma) = 0$, and thus is real analytic. Finally, the smooth solutions Γ of (4.20') are given by

$$\Gamma(y) = G \exp\left(\varphi(y)\right), \quad G \in \mathbb{R},$$

where $\varphi(y)$ is a real analytic function satisfying $\frac{\partial \varphi}{\partial y_1} = -\gamma_1^1$ and $\frac{\partial \varphi}{\partial y_2} = -\gamma_1^2$. Therefore, the systems Ξ_{pE} that are equivalent to Ξ'''_{pE} are parametrised by a real constant and two smooth functions γ^1 and γ^2 of the form (4.21) and satisfying (4.17'). If that constant is $G = 0$, then Ξ'_{pE} is equivalent to Ξ^0_{pE}; otherwise, if $G > 0$, resp. $G < 0$, then Ξ'_{pE} is equivalent to Ξ^+_{pE}, resp. Ξ^-_{pE}, see proposition below. ◆

The following proposition gives a canonical from of systems Ξ'''_{pE} depending on whether $c_0 \neq 0$ or $c_0 = 0$.

Proposition 4.5 (Canonical form of Ξ'''_{pE})**.** *Consider a pE-system Ξ_{pE} with structure functions $(\mu^k_i, \nu^k, \gamma^k)$ satisfying (4.16), (4.17), (4.18), (4.19), and (4.20). Then, it always admits one of the following canonical forms*

$$\Xi^{\pm}_{pE} \; : \; \dot{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ((w_1)^2 + (w_2)^2) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} w_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} w_2 + \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}, \quad \textit{or}$$

$$\Xi^{0}_{pE} \; : \; \dot{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ((w_1)^2 + (w_2)^2) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} w_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} w_2 + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Moreover, Ξ_{pE} is equivalent to Ξ^+_{pE} (resp. Ξ^-_{pE}) if and only if $\Gamma > 0$ (resp. $\Gamma < 0$) and to Ξ^0_{pE} if and only if $\Gamma \equiv 0$.

Proof. If Ξ_{pE} satisfies (4.16), (4.17), (4.18), (4.19), and (4.20), then it is equivalent to Ξ'''_{pE}. If $c_0 = 0$ then we have the canonical form Ξ^0_{pE}, otherwise use the following transformation $(w_1, w_2) = (\sqrt{|c_0|}\tilde{w}_1, \sqrt{|c_0|}\tilde{w}_2)$ and $(\tilde{z}, \tilde{y}_1, \tilde{y}_2) = \left(\frac{z}{|c_0|}, \frac{y_1}{\sqrt{|c_0|}}, \frac{y_2}{\sqrt{|c_0|}}\right)$ to obtain $\Xi^{\pm}_{pE}$.

Since we always have $\tilde{\Gamma} = \frac{\Gamma}{\lambda^2}$, clearly Ξ_{pE} (satisfying (4.16), (4.17), (4.18), (4.19), and (4.20)) is equivalent to Ξ^0_{pE} if and only if $\Gamma \equiv 0$ and is equivalent to Ξ^+_{pE} or Ξ^-_{pE} if and only if $\Gamma > 0$ or $\Gamma < 0$, respectively. ■

We finish this subsection by explaining how to get normal and canonical forms of p-elliptic submanifolds $\mathcal{S}_{pE}$. Recall that those submanifolds are given by an equation of the form $\dot{z} = \dot{y}^t Q(x)\dot{y} + b_1(x)\dot{y}_1 + b_2(x)\dot{y}_2 + c(x)$, with $Q = \left(\begin{smallmatrix} q_1 & q_2 \\ q_2 & q_3 \end{smallmatrix}\right)$ satisfying $\Delta_2 = \det(Q) = q_1 q_3 - (q_2)^2 > 0$. Hence a direct parametrisation of $\mathcal{S}_{pE}$, in terms of a control system, is given by

$$\Xi_{\mathcal{S}_{pE}} \; : \; \begin{cases} \dot{z} &= q_1(w_1^2) + 2q_2 w_1 w_2 + q_3 (w_2)^2 + b_1 w_1 + b_2 w_2 + c \\ \dot{y}_1 &= w_1 \\ \dot{y}_2 &= w_2 \end{cases},$$

where we can always assume that $q_1 > 0$. Observe that the above system $\Xi_{\mathcal{S}_{pE}}$ is not of the previously used form Ξ_{pE}. Indeed, $\Xi_{\mathcal{S}_{pE}}$ contains a mixed term $w_1 w_2$ and, moreover, the functions q_1 and q_3 may be different, so the vector field A cannot be identified. The following reparametrisation

$$\begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} = \begin{pmatrix} \sqrt{q_1} & \frac{q_2}{\sqrt{q_1}} \\ 0 & \sqrt{\frac{\Delta_2}{q_1}} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

transforms $\Xi_{\mathcal{S}_{pE}}$ into a system $\Xi^{\mathcal{S}}_{pE}$, of the form Ξ_{pE}, for which $A = \frac{\partial}{\partial z}$, the vector fields B_i depend on the functions q_i and b_i, and $C = c(x)\frac{\partial}{\partial z}$. The conditions of the previous results can be tested on $\Xi^{\mathcal{S}}_{pE}$ and the obtained normal forms give normal forms of $\mathcal{S}_{pE}$. More precisely, we have

Corollary 4.2 (Normal and canonical forms of p-elliptic submanifolds). *Consider a p-elliptic submanifold $\mathcal{S}_{pE} = \{\dot{z} = \dot{y}^t Q(x)\dot{y} + b_1(x)\dot{y}_1 + b_2(x)\dot{y}_2 + c(x)\}$ together with its parametrisation $\Xi^{\mathcal{S}}_{pE}$. The following statements hold:*

(i) If $\Xi^{\mathcal{S}}_{pE}$ is equivalent to Ξ^{d}_{pE}, then $\mathcal{S}_{pE}$ is equivalent to

$$\mathcal{S}^d_{pE} = \{\lambda_1(x)(\dot{y}_1)^2 + \lambda_2(x)(\dot{y}_2)^2 + b_1(x)\dot{y}_1 + b_2(x)\dot{y}_2 + c(x)\},$$

where $\lambda_i > 0$.

(ii) If $\Xi^{\mathcal{S}}_{pE}$ is equivalent to Ξ'_{pE}, then $\mathcal{S}_{pE}$ is equivalent to

$$\mathcal{S}'_{pE} = \{(\dot{y}_1)^2 + (\dot{y}_2)^2 + b_1(x)\dot{y}_1 + b_2(x)\dot{y}_2 + c(x)\}.$$

(iii) If $\Xi^{\mathcal{S}}_{pE}$ is equivalent to Ξ''_{pE}, then $\mathcal{S}_{pE}$ is equivalent to

$$\mathcal{S}''_{pE} = \{(\dot{y}_1)^2 + (\dot{y}_2)^2 + c(x)\}.$$

(iv) If $\Xi^{\mathcal{S}}_{pE}$ is equivalent to Ξ'''_{pE}, then $\mathcal{S}_{pE}$ is equivalent to $\mathcal{S}'''_{pE} = \{(\dot{y}_1)^2 + (\dot{y}_2)^2 + c\}$, with $c \in \mathbb{R}$ and, moreover, c can always be normalised to either $c = 0$ or $c = \pm 1$.

Remark. All four statements *(i)* to *(iv)* are, actually, «if and only if» statements but we presented them as implications that show how equivalence of control systems allows to solve the original problem of equivalence of pE-submanifolds. ◆

Example. If $q_2 \equiv 0$, then the conditions of the first above statements can readily be enunciated. In that case, the system $\Xi^{\mathcal{S}}_{pE}$ is given by the following vector fields:

$$A = \frac{\partial}{\partial z},\ B_1 = \begin{pmatrix} b_1/\sqrt{q_1} \\ 1/\sqrt{q_1} \\ 0 \end{pmatrix},\ B_2 = \begin{pmatrix} b_2/\sqrt{q_3} \\ 0 \\ 1/\sqrt{q_3} \end{pmatrix},\ \text{and } C = c\frac{\partial}{\partial z}.$$

Structure functions μ_i^j, ν^k, γ^k attached to this frame are given by:

$$\begin{aligned}
&\mu_1^0 = \frac{1}{\sqrt{q_1}}\frac{\partial b_1}{\partial z},\ \mu_1^1 = -\frac{1}{2q_1}\frac{\partial q_1}{\partial z},\ \mu_1^2 = 0,\\
&\mu_2^0 = \frac{1}{\sqrt{q_3}}\frac{\partial b_2}{\partial z},\ \mu_2^1 = 0,\ \mu_1^2 = -\frac{1}{2q_3}\frac{\partial q_3}{\partial z},\\
&\nu^0 =,\ \nu^1,\ \nu^2\\
&\gamma^0 = c,\ \gamma^1 = \gamma^2 = 0.
\end{aligned}$$

Hence, $\mathcal{S}_{pE}$ is equivalent to $\mathcal{S}'_{pE}$ if and only if, $-\frac{1}{2q_1}\frac{\partial q_1}{\partial z} = -\frac{1}{2q_3}\frac{\partial q_3}{\partial z}$, equivalently, $\frac{\partial}{\partial z}\left(\frac{q_1}{q_3}\right) = 0$. ●

The normal form $\mathcal{S}^d_{pE}$ describes the diagonalisation of the matrix Q and for the normal form $\mathcal{S}'_{pE}$, the matrix Q is fully normalised. Then with $\mathcal{S}''_{pE}$, we additionally normalised $b = (b_1, b_2)$ and finally with $\mathcal{S}'''_{pE}$ we describe p-elliptic submanifolds with no functional parameters, i.e. which do not depend on the point $x \in \mathcal{X}$.

In this subsection we studied the classification problem of nonlinear p-elliptic systems under the action of the group of feedback. Our classification includes several normal forms and canonical forms. The conditions that we proposed are checkable in terms of algebraic and differential relations between structure functions attached to the p-elliptic structure of the system.

2 Study of p-Hyperbolic systems

We now turn to the study of the equivalence to a p-hyperbolic system represented by the normal form

$$\Sigma_{1,1} \ : \left\{ \begin{array}{ll} \dot{x} & = A(x)(w_1^2 - w_2^2) + B(x)w + C(x) \\ \dot{w} & = u \end{array} \right. ,$$

where A, $B = (B_1, B_2)$, and C are smooth vector fields on a 3-dimensional manifold $\mathcal{X} = \mathcal{M}/\mathcal{D}^0$. However, it will be more convenient to work with the following feedback equivalent form, obtained by the change of coordinates $\tilde{w}_1 = \frac{1}{2}(w_1 + w_2)$ and $\tilde{w}_2 = \frac{1}{2}(w_1 - w_2)$, which is also denoted Σ_{pH} (where we deleted the tildes),

$$\Sigma_{pH} \ : \left\{ \begin{array}{ll} \dot{x} & = A(x)w_1w_2 + B(x)w + C(x) \\ \dot{w} & = u \end{array} \right. .$$

The p-hyperbolic case is, in some sense, easier to analyse than the p-elliptic case, because due to the last Σ_{pH}-form the system is affine separately with respect to both w_1 and w_2 (and not quadratic as in the p-elliptic case). In this section whenever we refer to assumption (A3), we always mean

$(\mathrm{A3})_{\mathrm{pH}}$ $\operatorname{sgn}(\Omega_\omega) = (1, 1)$.

In the following subsections we will, first, give a complete characterisation of p-hyperbolic systems in terms of checkable algebraic and differential relations between structure functions attached to control-affine systems and, second, working within the class of p-hyperbolic systems we will give normal and canonical forms of p-hyperbolic systems.

2.1 Characterisation of p-hyperbolic systems

In this subsection, we will, first, introduce general objects used in the statements of the results and, second, we will fully characterise p-hyperbolic systems Σ_{pH}, when possible, we will give a geometric interpretation of our conditions. Those geometric interpretations permit to characterise of a more specific class of p-hyperbolic systems (see Theorem 4.11).

Consider a control-affine system $\Sigma : \dot{\xi} = f + g_1u_1 + g_2u_2$ on a 5-dimensional manifold $\mathcal{M}$ and for which we define the following distributions

$$\mathcal{D}^0 = \operatorname{span}\{g_1, g_2\}, \quad \text{and} \quad \mathcal{D}^1 = \operatorname{span}\{g_1, g_2, \operatorname{ad}_f g_1, \operatorname{ad}_f g_2\}.$$

Recall that assumptions (A1) and (A2) imply that $\mathcal{D}^0$ is involutive and of constant rank 2, and that $\mathcal{D}^1$ is of constant rank 4, moreover, $(\mathrm{A3})_{\mathrm{pH}}$ implies that $\mathcal{D}^1$ is not involutive.

Definition 4.8 (Weak isotropic frame)**.** We say that a pair (g_1, g_2) is a *weak isotropic frame* of $\mathcal{D}^0 = \operatorname{span}\{g_1, g_2\}$, shortly *WIF*, if

$$[g_1, \operatorname{ad}_f g_1] = [g_2, \operatorname{ad}_f g_2] = 0 \mod \mathcal{D}^1, \quad \text{and} \quad [g_1, \operatorname{ad}_f g_2] - [g_2, \operatorname{ad}_f g_1] = 0 \mod \mathcal{D}^1.$$

Notice that due to the Jacobi identity, the second condition is always satisfied but we included it here for consistence with the previous section and the results of the following chapter. Given a weak isotropic frame, we introduce the structure functions $\mu^j_{i,i}$ and μ^j for $i,j = 1,2$

$$
\begin{aligned}
[g_i, \mathrm{ad}_f g_i] \quad &= \mu^1_{i,i}\mathrm{ad}_f g_1 + \mu^2_{i,i}\mathrm{ad}_f g_2 \mod \mathcal{D}^0, \\
[g_1, \mathrm{ad}_f g_2] - [g_2, \mathrm{ad}_f g_1] \quad &= \mu^1 \mathrm{ad}_f g_1 + \mu^2 \mathrm{ad}_f g_2 \mod \mathcal{D}^0.
\end{aligned}
\tag{4.22}
$$

The structure functions μ^j coincide with the structure functions (ν^1, ν^2), defined by $[g_1, g_2] = \nu^1 g_1 + \nu^2 g_2$; indeed, by the Jacobi identity, we have $\mu^j = \nu^j$.

Proposition 4.6 (Existence and properties of weak isotropic frames)**.**

(i) *Under assumptions (A1), (A2), and (A3)$_{pH}$, there exists a weak isotropic frame.*

(ii) *If $(\tilde{g}_1, \tilde{g}_2)$ is a WIF then $(g_1, g_2) = (\tilde{g}_1, \tilde{g}_2)\beta$ is also a WIF if and only if either $\beta = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}$ or $\beta = \begin{pmatrix} 0 & \beta_1 \\ \beta_2 & 0 \end{pmatrix}$, where β_i are smooth functions on $\mathcal{M}$ satisfying $\beta_i(\cdot) \neq 0$.*

(iii) *Under the feedback $(g_1, g_2) = (\beta_1\tilde{g}_1, \beta_2\tilde{g}_2)$, the structure functions $\mu^j_{i,i}$ and μ^j of a weak isotropic frame (g_1, g_2) and $\tilde{\mu}^j_{i,i}$ and $\tilde{\mu}^j$ of $(\tilde{g}_1, \tilde{g}_2)$ are related by*

$$
\begin{aligned}
\mu^1_{1,1} &= (\beta_1)\tilde{\mu}^1_{1,1} + \mathrm{L}_{\tilde{g}_1}(\beta_1), & \mu^2_{1,1} &= \frac{(\beta_1)^2}{\beta_2}\tilde{\mu}^2_{1,1}, \\
\mu^1_{2,2} &= \frac{(\beta_2)^2}{\beta_1}\tilde{\mu}^1_{2,2}, & \mu^2_{2,2} &= (\beta_2)\tilde{\mu}^2_{2,2} + \mathrm{L}_{\tilde{g}_2}(\beta_2), \\
\mu^1 &= \beta_2\tilde{\mu}^1 - \frac{\beta_2}{\beta_1}\mathrm{L}_{\tilde{g}_2}(\beta_1), & \mu^2 &= \beta_1\tilde{\mu}^2 + \frac{\beta_1}{\beta_2}\mathrm{L}_{\tilde{g}_1}(\beta_2),
\end{aligned}
\tag{4.23}
$$

and the transformation $f \mapsto f + g\alpha$ does not change them.

Remark (Restriction of the feedback action). In all formulae below, we suppose that the feedback β is diagonal (i.e. of the first type of statement *(ii)* of the above proposition), clearly the anti-diagonal β requires permuting the fields g_1 and g_2 which does not change the presented results. ◆

Proof.

(i) Consider a control-affine system Σ given by vector fields f and $g = (g_1, g_2)$. By Proposition 3.1, there exists a feedback β_1, such that for the new frame $\bar{g} = g\beta_1$ we have $\bar{\Omega}_\omega = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Compose this feedback with $\beta_2 = \begin{pmatrix} 1 & \mu \\ 1 & -\mu \end{pmatrix}$, where μ is an arbitrary non-vanishing function, to obtain $\tilde{\Omega}_\omega = \begin{pmatrix} 0 & 2\mu \\ 2\mu & 0 \end{pmatrix}$ for the frame $\tilde{g} = g\beta_1\beta_2$. Clearly $\tilde{g}$ is a weak isotropic frame.

(ii) Assume that (g_1, g_2) and $(\tilde{g}_1, \tilde{g}_2)$ are two weak isotropic frames related by $\tilde{g} = g\beta$. By relation (3.1) we have $\tilde{\Omega}_\omega = \beta^t\Omega_\omega\beta$ with $\tilde{\Omega}_\omega = \begin{pmatrix} 0 & \tilde{\mu} \\ \tilde{\mu} & 0 \end{pmatrix}$ and $\Omega_\omega = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}$. By a direct computation, we obtain that β takes the required forms.

(iii) See calculations in Appendix 4.D.

■

Statements *(i)* and *(ii)* of the above proposition show that in the p-hyperbolic case, under (A1), (A2), and (A3)$_{\mathrm{pH}}$, we can attach to Σ two well-defined mutually transversal subdistributions of rank 1 of $\mathcal{D}^0$. Moreover, given any WIF (g_1, g_2), those distributions are defined as $\mathcal{G}_1 = \operatorname{span}\{g_1\}$ and $\mathcal{G}_2 = \operatorname{span}\{g_2\}$ and do not depend on the choice of that WIF.

Under basic assumptions (A1), (A2), and $(A3)_{\mathrm{pH}}$ there always exists a weak isotropic frame and we are now going to reinforce this notion which will turn out to be the key of the characterisation of p-hyperbolic systems.

Definition 4.9 (Strong isotropic frame)**.** We say that a pair (g_1, g_2) is a *strong isotropic frame* of $\mathcal{D}^0 = \operatorname{span}\{g_1, g_2\}$, shortly SIF, if

$$[g_1, \mathrm{ad}_f g_1] = [g_2, \mathrm{ad}_f g_2] = 0 \mod \mathcal{D}^0, \quad \text{and} \quad [g_1, \mathrm{ad}_f g_2] - [g_1, \mathrm{ad}_f g_2] = 0 \mod \mathcal{D}^0.$$

In other words, a strong isotropic frame is a weak isotropic frame with all structure functions satisfying $\mu^j_{i,i} = 0$ and $\mu^j = 0$.

Proposition 4.7 (Properties of strong isotropic frames)**.**

(i) Any p-hyperbolic system Σ_{pH} possesses a strong isotropic frame,

(ii) If $(\tilde{g}_1, \tilde{g}_2)$ is a strong isotropic frame, then $(g_1, g_2) = (\beta_1\tilde{g}_1, \beta_2\tilde{g}_2)$ is a weak isotropic frame whose structure functions $\mu^j_{i,i}$ and μ^j satisfy

$$\mu^2_{1,1} = \mu^1_{2,2} = 0, \quad \text{and}$$
$$\mu^1_{1,1} = \mathrm{L}_{\tilde{g}_1}(\beta_1), \ \mu^2_{2,2} = \mathrm{L}_{\tilde{g}_2}(\beta_2), \ \mu^1 = -\frac{\beta_2}{\beta_1}\mathrm{L}_{\tilde{g}_2}(\beta_1), \ \mu^2 = \frac{\beta_1}{\beta_2}\mathrm{L}_{\tilde{g}_1}(\beta_2).$$

(iii) If $(\tilde{g}_1, \tilde{g}_2)$ is a strong isotropic frame, then $\tilde{g}_1$ and $\tilde{g}_2$ are commuting vector fields.

Proof.

(i) Recall that Σ_{pH} is given by the vector fields $(g_1, g_2) = \left(\frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}\right)$ and $f = A(x)w_1w_2 + B(x)w + C(x) \mod \mathcal{D}^0$. Then, clearly, (g_1, g_2) form a strong isotropic frame of Σ_{pH}.

(ii) Assume that $(\tilde{g}_1, \tilde{g}_2)$ is a strong isotropic frame so it is a WIF satisfying $\tilde{\mu}^j_{i,i} = \tilde{\mu}^j = 0$. Then, $(g_1, g_2) = (\beta_1\tilde{g}_1, \beta_2\tilde{g}_2)$ is also a WIF and applying formula (4.23) we obtain the required relations for the structure function $\mu^j_{i,i}$ and μ^j of (g_1, g_2).

(iii) Assume that $(\tilde{g}_1, \tilde{g}_2)$ is a strong isotropic frame, i.e. $\tilde{\mu}^j_{i,i} = \tilde{\mu}^j = 0$. Express $[\tilde{g}_1, \tilde{g}_2] = \tilde{\nu}^1\tilde{g}_1 + \tilde{\nu}^2\tilde{g}_2$, then by the Jacobi identity applied to $[f, [\tilde{g}_1, \tilde{g}_2]]$, we obtain $\tilde{\nu}^j = \tilde{\mu}^j = 0$ for $j = 1, 2$.

■

From statement *(ii)* of the above proposition, we see that if Σ is feedback equivalent to Σ_{pH} (and thus possesses a SIF due to statement *(i)*), then the structure functions of any its WIF have to satisfy

$$\mu^2_{1,1} = 0, \quad \text{and} \quad \mu^1_{2,2} = 0. \tag{4.24}$$

Moreover, observe that a feedback β (in fact the logarithm of β_1 and β_2) assuring the passage from a WIF (g_1, g_2) into a SIF $(\tilde{g}_1, \tilde{g}_2)$ satisfies two systems of two first order partial differential equations:

$$\left\{ \begin{array}{ll} \mathrm{L}_{g_1}\left(\ln(\beta_1)\right) & = \mu^1_{1,1} \\ \mathrm{L}_{g_2}\left(\ln(\beta_1)\right) & = -\mu^1 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll} \mathrm{L}_{g_1}\left(\ln(\beta_2)\right) & = \mu^2 \\ \mathrm{L}_{g_2}\left(\ln(\beta_2)\right) & = \mu^2_{2,2} \end{array} \right. . \tag{4.25}$$

Integrability conditions for those systems give relations between the structure functions that are necessary for the existence of a SIF (recall that $\mu^j = \nu^j$ and that $[g_1, g_2] = \nu^1 g_1 + \nu^2 g_2$):

$$\begin{array}{ll} \mathrm{L}_{g_1}\left(\mu^1\right) + \mathrm{L}_{g_2}\left(\mu^1_{1,1}\right) & = -\nu^1\mu^1_{1,1} + \nu^2\mu^1, \quad \text{and} \\ \mathrm{L}_{g_1}\left(\mu^2_{2,2}\right) - \mathrm{L}_{g_2}\left(\mu^2\right) & = \nu^1\mu^2 + \nu^2\mu^2_{2,2}. \end{array} \tag{4.26}$$

Observe that relations (4.24) and (4.26) are algebraic and differential, respectively, and thus can be explicitly tested on any given weak isotropic frame.

The following theorem shows, first, that those conditions are also sufficient for the existence of a SIF and, second, that the existence of a SIF fully characterises p-hyperbolic systems Σ_{pH}.

Theorem 4.10 (Characterisation of p-hyperbolic systems). *Consider a control-affine system Σ satisfying assumptions(A1), (A2), and $(A3)_{pH}$. Then the following statements are equivalent.*

(pH1) *Σ is feedback equivalent to Σ_{pH};*

(pH2) *For any weak isotropic frame (g_1, g_2) of Σ, the structure functions $\mu^k_{i,i}$ and μ^k satisfy (4.24) and, moreover, the systems given by (4.25) have solutions;*

(pH3) *For any weak isotropic frame (g_1, g_2) of Σ, the structure functions $\mu^k_{i,i}$ and μ^k satisfy (4.24) and (4.26);*

(pH4) *There exists a strong isotropic frame of Σ;*

As in the p-elliptic case, we see that the existence of a *strong* frame is the crucial condition in the characterisation of p-hyperbolic system.

Remark (Geometry of p-hyperbolic systems). The geometry of the existence of a strong isotropic frame is the following. For any weak isotropic frame the following distributions

$$\mathcal{D}^0 = \operatorname{span}\{g_1, g_2\}, \quad \mathcal{E}^1 = \operatorname{span}\{g_1, \operatorname{ad}_f g_1\}, \quad \text{and} \quad \mathcal{E}^2 = \operatorname{span}\{g_2, \operatorname{ad}_f g_2\}$$

are involutive, $\mathcal{E}_1$ and $\mathcal{E}_2$ modulo $\mathcal{D}^0$. For each one of them, using a feedback $\beta = (\beta_1, \beta_2)$, we can construct generators that commute ($\mathcal{E}_1$ and $\mathcal{E}_2$ modulo $\mathcal{D}^0$). The existence of a strong isotropic frame implies that this construction can be done for all three distributions simultaneously, i.e. there exists a feedback $\beta = (\beta_1, \beta_2)$ such that $[\tilde{g}_1, \tilde{g}_2] = [\tilde{g}_1, \operatorname{ad}_f \tilde{g}_1] = [\tilde{g}_2, \operatorname{ad}_f \tilde{g}_2] = 0$. ◆

Proof. We will show (pH1)⇒(pH2)⇒(pH3)⇒(pH4)⇒(pH1).
(pH1)⇒(pH2)⇒(pH3). It is the analysis performed above the theorem.
(pH3)⇒(pH4). Consider a control-affine system Σ given by vector fields f and (g_1, g_2) and assume that (g_1, g_2) is a weak isotropic frame whose structure functions

$\mu_{i,i}^j$ and μ^j satisfy (4.24) and (4.26). Condition (4.26) describe the integrability conditions of the systems (4.25), thus smooth solutions β_1 and β_2 exists and observe that necessarily such solutions satisfy $\beta_i(\cdot) \neq 0$. Let β_1 and β_2 be any smooth solutions of systems (4.25) and apply the feedback $\tilde{f} = f$ and $(\tilde{g}_1, \tilde{g}_2) = \left(\frac{1}{\beta_1} g_1, \frac{1}{\beta_2} g_2\right)$. We claim that the pair $(\tilde{g}_1, \tilde{g}_2)$ is a strong isotropic frame (see calculations in Appendix 4.E).
(pH4)⇒(pH1). Assume that the pair (g_1, g_2) is a SIF of Σ, recall from statement *(iii)* of Proposition 4.7 that such pair satisfies $[g_1, g_2] = 0$. Therefore, introduce coordinates $(x, w) = (z, y_1, y_2, w_1, w_2) = \phi(\xi)$ such that $\phi_* g_1 = \frac{\partial}{\partial w_1}$ and $\phi_* g_2 = \frac{\partial}{\partial w_2}$. After applying a suitable feedback, $f \mapsto f + \alpha_1 g_1 + \alpha_2 g_2$, the system Σ takes the form

$$\begin{cases} \dot{x} &= \mathsf{f}(x, w) \\ \dot{w}_1 &= u_1 \\ \dot{w}_2 &= u_2 \end{cases},$$

for which $(g_1, g_2) = \left(\frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}\right)$ is a strong isotropic frame. By definition of strong isotropic frames, we have the following conditions on f

$$\frac{\partial^2 \mathsf{f}}{\partial w_1^2} = \frac{\partial^2 \mathsf{f}}{\partial w_2^2} = 0.$$

Solutions of those equations admit the following closed form,

$$\mathsf{f}(x, w) = A(x) w_1 w_2 + B_1(x) w_1 + B_2(x) w_2 + C(x),$$

where A, B_1, B_2, and C are smooth vector fields on $\mathcal{M}/\mathcal{D}^0$. For that form we have $\mathcal{D}^1 = \operatorname{span}\{A w_2 + B_1, A w_1 + B_2\} \mod \mathcal{D}^0$ which, by assumption (A2), is of constant rank 4 and since we have preserved the signature of Ω_ω in all our computations we conclude that $A \notin \mathcal{D}^1$ and thus we have $A \wedge B_1 \wedge B_2 \neq 0$. ∎

The characterisation of p-hyperbolic systems require two conditions. The first one (4.24) is purely algebraic and is related to the p-hyperbolic structure of Σ_{pH}; the second one (4.26) is differential and asserts that there exists a feedback β that transforms a WIF into a SIF. In the following two remarks we are going to give an interpretation of those two conditions.

Remark (Geometric interpretation of relation (4.24)). We already observed that any WIF identifies two subdistributions $\mathcal{G}_1 = \operatorname{span}\{g_1\}$ and $\mathcal{G}_2 = \operatorname{span}\{g_2\}$ of rank 1 of $\mathcal{D}^0$. To them we add two subdistributions of $\mathcal{D}^1$:

$$\mathcal{A}_1 = \mathcal{D}^0 + [f, \mathcal{G}_1], \quad \text{and} \quad \mathcal{A}_2 = \mathcal{D}^0 + [f, \mathcal{G}_2]. \tag{4.27}$$

Notice that, by assumption (A2), those distributions are of constant rank 3 and they are not involutive (otherwise Ω_ω would be 0), and they are invariant under feedback transformations $f \mapsto f + g\alpha$ and $(\tilde{g}_1, \tilde{g}_2) = (\beta_1 g_1, \beta_2 g_2)$. Then, condition (4.24) is equivalent to

$$\mathcal{G}_1 = \mathcal{C}(\mathcal{A}_1), \quad \text{and} \quad \mathcal{G}_2 = \mathcal{C}(\mathcal{A}_2), \tag{4.28}$$

where $\mathcal{C}(\mathcal{A}_i)$ denotes the characteristic distribution of $\mathcal{A}_i$. Indeed, clearly, (4.28) implies (4.24). Conversely, condition (4.24) implies that $\mathcal{G}_i \subset \mathcal{C}(\mathcal{A}_i)$, for $i = 1, 2$ and we show now that the characteristic distribution of $\mathcal{A}_1$ is of rank 1. Assume that the vector field $v = \gamma^1 g_2 + \gamma^2 \mathrm{ad}_f g_1 \in \mathcal{A}_1$, where γ^i are smooth functions, is also characteristic, i.e. $v \in \mathcal{C}(\mathcal{A}_1)$. Then, we have

$$\left[\gamma^1 g_2 + \gamma^2 \mathrm{ad}_f g_1, \mathrm{ad}_f g_1\right] = \gamma^1 \left[g_2, \mathrm{ad}_f g_1\right] \mod \mathcal{A}_1$$

but by (A3)$_{\mathrm{pH}}$ we have $[g_2, \mathrm{ad}_f g_1] \notin \mathcal{D}^1$ and thus we must have $\gamma^1 = 0$. Similarly, $[\gamma^2 \mathrm{ad}_f g_1, g_2] \in \mathcal{A}_1$ implies that $\gamma^2 = 0$. Therefore if $v \in \mathcal{C}(\mathcal{A}_1)$, then $v = 0$; hence the characteristic distribution of $\mathcal{A}_1$ has rank 1 and is equal to $\mathcal{G}_1$. An analogous reasoning applied to $\mathcal{A}_2$ shows that $\mathcal{C}(\mathcal{A}_2)$ is of rank 1, and thus equal to $\mathcal{G}_2$. ◆

Remark (Interpretation of relation (4.26)). For any WIF we consider the systems of first order partial differential equations given by (4.25). We can always solve two (out of four) equations $\mathrm{L}_{g_2}(\ln(\beta_1)) = -\mu^1$ and $\mathrm{L}_{g_1}(\ln(\beta_2)) = \mu^2$. Taking β_1 and β_2 as smooth solutions of those two equations yields a new frame $(\tilde{g}_1, \tilde{g}_2) = \left(\frac{1}{\beta_1} g_1, \frac{1}{\beta_2} g_2\right)$ which is commutative but not necessarily a SIF. In that frame relation (4.26) reads $\mathrm{L}_{\tilde{g}_2}\left(\tilde{\mu}^1_{1,1}\right) = \mathrm{L}_{\tilde{g}_1}\left(\tilde{\mu}^2_{2,2}\right) = 0$. Therefore, if a commutative WIF is equivalent to a SIF, then 4 structure functions are normalised, namely $\mu^1 = \mu^2 = \mu^2_{1,1} = \mu^1_{2,2} = 0$, and the last two $\mu^1_{1,1}$ and $\mu^2_{2,2}$ depend on one of the w variables only. ◆

The distributions $\mathcal{A}_i$ defined by (4.27) carry some informations about the vector fields A, B_1, and B_2 of the resulting p-hyperbolic system. In the following we will give a geometric characterisation of the following normal form

$$\Sigma'_{pH} : \begin{cases} \dot{z} &= w_1 w_2 + c_0(x) \\ \dot{y}_i &= w_i + c_i(x) \\ \dot{w}_i &= u_i \end{cases} .$$

This characterisation will be further investigated in the following subsection.

Theorem 4.11 (Geometric characterisation of Σ'_{pH}). *Under assumptions (A1), (A2), and (A3)$_{pH}$, the system Σ is feedback equivalent to Σ'_{pH} if and only if, for $i = 1, 2$,*

- *(i) The distributions $\mathcal{G}_i$ satisfy $\mathcal{G}_i = \mathcal{C}(\mathcal{A}_i)$,*
- *(ii) The distributions $\mathcal{A}_i$ have the growth vector $(3, 4, 4)$.*

Remark (Interpretation of the conditions). Observe that for a general p-hyperbolic system Σ_{pH} the distributions $\mathcal{A}_i$ read

$$\mathcal{A}_1 = \mathrm{span}\left\{A w_2 + B_1\right\} + \mathcal{D}^0 \quad \text{and} \quad \mathcal{A}_2 = \mathrm{span}\left\{A w_1 + B_2\right\} + \mathcal{D}^0.$$

Moreover, we have then $\mathcal{A}_i + [\mathcal{A}_i, \mathcal{A}_i]$ is of constant rank 4, since $A \wedge B_1 \wedge B_2 \neq 0$, and $\mathcal{A}_i + [\mathcal{A}_i, \mathcal{A}_i] = \mathcal{A}_i + \mathrm{span}\{A\} = \mathrm{span}\{A, B_i\} + \mathcal{D}^0$. For that system, condition *(i)* is always fulfilled, and condition *(ii)* additionally requires that $\mathcal{A}_i + [\mathcal{A}_i, \mathcal{A}_i]$ is involutive, that is,

$$[A, B_i] = 0 \mod \mathcal{D}^0 + \mathrm{span}\{A, B_i\}.$$

In other words, the distributions $\mathcal{A}_i$ are not involutive but in a *minimal* way, that is, only one Lie bracket sticks out of $\mathcal{A}_i$ and if we add it to $\mathcal{A}_i$ we get $\bar{\mathcal{A}}_i$ (the involutive closure of $\mathcal{A}_i$), involutive of rank 4. In the following subsection, see in particular statement *(v)* of Proposition 4.9, we will give another interpretation of that condition. ◆

Proof. Clearly, the distributions $\mathcal{G}_i$ and $\mathcal{A}_i$ do not depend on feedback transformations $f \mapsto f + g\alpha$ and $(g_1, g_2) = (\beta_1\tilde{g}_1, \beta_2\tilde{g}_2)$. Moreover, Σ'_{pH} satisfies *(i)* and *(ii)*. Indeed, for Σ'_{pH}, we have $\mathcal{A}_1 = \mathrm{span}\left\{\frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}, w_2\frac{\partial}{\partial z} + \frac{\partial}{\partial y_1}\right\}$ and $\mathcal{A}_2 = \mathrm{span}\left\{\frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}, w_1\frac{\partial}{\partial z} + \frac{\partial}{\partial y_2}\right\}$ leading to $\mathcal{A}_i + [\mathcal{A}_i, \mathcal{A}_i] = \mathcal{A}_i + \mathrm{span}\left\{\frac{\partial}{\partial z}\right\}$, for $i = 1, 2$, which are of constant rank 4 and involutive. This concludes the necessity part of the proof.

Conversely, assume that Σ satisfies *(i)* and *(ii)*. Denote by $\bar{\mathcal{A}}_i$ the involutive closure of $\mathcal{A}_i$, which by assumption is of constant corank one. By the Frobenius theorem, the annihilator of $\bar{\mathcal{A}}_1$ (resp. $\bar{\mathcal{A}}_2$) is given the differential $\mathrm{d}\tilde{y}_2$ (resp. $\mathrm{d}\tilde{y}_1$) of a real function $\tilde{y}_2$ (resp. $\tilde{y}_1$). Introduce coordinates $\tilde{x} = (\tilde{z}, \tilde{y}_1, \tilde{y}_2)$ such that $\mathrm{d}\tilde{x} \in \mathrm{ann}\,(\mathcal{D}^0)$, $\mathrm{d}\tilde{y}_1 \in \mathrm{ann}\left(\bar{\mathcal{A}}_2\right)$, and $\mathrm{d}\tilde{y}_2 \in \mathrm{ann}\left(\bar{\mathcal{A}}_1\right)$. Complete $\tilde{x}$ with $\tilde{w} = (\tilde{w}_1, \tilde{w}_2)$ to a coordinate system $(\tilde{x}, \tilde{w})$ such that $\mathrm{d}\tilde{w}_1 \in \mathrm{ann}\,(\mathcal{G}_2)$ and $\mathrm{d}\tilde{w}_2 \in \mathrm{ann}\,(\mathcal{G}_1)$. In those coordinates, the system Σ takes the form,

$$\begin{cases} \dot{\tilde{z}} & = f_0(\tilde{x}, \tilde{w}) \\ \dot{\tilde{y}}_1 & = f_1(\tilde{x}, \tilde{w}) \\ \dot{\tilde{y}}_2 & = f_2(\tilde{x}, \tilde{w}) \\ \dot{\tilde{w}}_1 & = c_1(\tilde{x}, \tilde{w})u_1 \\ \dot{\tilde{w}}_2 & = c_2(\tilde{x}, \tilde{w})u_2 \end{cases},$$

with $f = f_0\frac{\partial}{\partial \tilde{z}} + f_1\frac{\partial}{\partial \tilde{y}_1} + f_2\frac{\partial}{\partial \tilde{y}_2}$, $g_1 = c_1\frac{\partial}{\partial \tilde{w}_1}$, and $g_2 = c_2\frac{\partial}{\partial \tilde{w}_2}$. Since $\langle \mathrm{d}\tilde{y}_1, \mathrm{ad}_f g_2\rangle = \langle \mathrm{d}\tilde{y}_2, \mathrm{ad}_f g_1\rangle = 0$, we have $f_1 = f_1(\tilde{x}, \tilde{w}_1)$ and $f_2 = f_2(\tilde{x}, \tilde{w}_2)$. Moreover, assumption (A2) implies that $\frac{\partial f_1}{\partial \tilde{w}_1} \neq 0$ and $\frac{\partial f_2}{\partial \tilde{w}_2} \neq 0$. Thus, introducing new coordinates $\bar{w}_i = f_i(\tilde{x}, \tilde{w}_i)$ (observe that this transformation does not affect the distributions $\mathcal{G}_i$ and thus the distributions $\mathcal{A}_i$ either) yields

$$\mathcal{A}_1 = \mathrm{span}\left\{\frac{\partial}{\partial \bar{w}_1}, \frac{\partial}{\partial \bar{w}_2}, \frac{\partial}{\partial \tilde{y}_1} + \frac{\partial f_0}{\partial \bar{w}_1}\frac{\partial}{\partial \tilde{z}}\right\} \text{ and } \mathcal{A}_2 = \mathrm{span}\left\{\frac{\partial}{\partial \bar{w}_1}, \frac{\partial}{\partial \bar{w}_2}, \frac{\partial}{\partial \tilde{y}_2} + \frac{\partial f_0}{\partial \bar{w}_2}\frac{\partial}{\partial \tilde{z}}\right\}.$$

Since g_1 (resp. g_2) is a characteristic vector field for $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) we have $[g_1, \mathrm{ad}_f g_1] \in \mathcal{A}_1$ and $[g_2, \mathrm{ad}_f g_2] \in \mathcal{A}_2$ implying that $\frac{\partial^2 f_0}{\partial \bar{w}_1^2} = \frac{\partial^2 f_0}{\partial \bar{w}_2^2} = 0$. Therefore we have $f_0 = a(\tilde{x})\bar{w}_1\bar{w}_2 + \tilde{b}_1(\tilde{x})\bar{w}_1 + \tilde{b}_2(\tilde{x})\bar{w}_2 + \tilde{c}(\tilde{x})$, notice that $a(\tilde{x}_0) \neq 0$ otherwise we would have $\dim(\bar{\mathcal{A}}_i(\tilde{x}_0)) = 3$. Introduce a new coordinate system $x = (z, y_1, y_2) = (\psi(\tilde{x}), \tilde{y}_1, \tilde{y}_2)$ such that $\dot{z} = \bar{w}_1\bar{w}_2 + \bar{b}_1(x)\bar{w}_1 + \bar{b}_2(x)\bar{w}_2 + \bar{c}(x)$, for some new functions $\bar{b}_1$, $\bar{b}_2$, and $\bar{c}$, then introducing the following change $(w_1, w_2) = (\bar{w}_1 + \bar{b}_2, \bar{w}_2 + \bar{b}_1)$ produces, after applying a suitable feedback along the last two components, the normal form Σ'_{pH}. ∎

In this subsection, we fully characterised the feedback equivalence of a control-affine system Σ to a p-hyperbolic system Σ_{pH} and thus we solved the problem of characterisation of p-hyperbolic submanifolds $\mathcal{S}_{pH}$ (of $T\mathcal{X}$). Our characterisation involves the construction of weak and strong isotropic frames of $\mathcal{D}^0$ and the existence of a strong isotropic frame is the key of that characterisation. The conditions we developed are necessary and sufficient and are checkable in terms of algebraic and differential relations between structure functions attached to the system Σ. Finally, we identified a geometry of a subclasses of p-hyperbolic systems and we used that geometry to characterise the p-hyperbolic system Σ'_{pH}.

In the following subsection, we will work inside the class of p-hyperbolic systems and we will show multiple classification results, involving the construction of several normal and canonical forms.

2.2 Classification of p-hyperbolic systems

We now investigate the problem of classification of p-hyperbolic submanifolds $\mathcal{S}_{pH}$ of $T\mathcal{X}$. This problem is dealt with under the classification of its first prolongations defined by

$$\Xi_{pH} \,:\, \dot{x} = A(x)w_1w_2 + B(x)w + C(x),$$

where A, $B = (B_1, B_2)$, and C are smooth vector fields on $\mathcal{X}$. Moreover, we assume that $A \wedge B_1, \wedge B_2 \neq 0$ in a neighbourhood of x_0. The first prolongation Ξ_{pH} is treated as a control-nonlinear system with state $x \in \mathcal{X}$ and controls $w = (w_1, w_2)$ and is represented by the 4-tuple of vector fields (A, B_1, B_2, C). We will describe several orbits of Ξ_{pH} under the action of feedback transformations $\tilde{x} = \phi(x)$ and $w = \psi(x, \tilde{w})$. First of all we have the following characterisation of admissible feedback transformations.

Proposition 4.8 (Equivalence of p-hyperbolic systems)**.**

(i) If two p-hyperbolic systems $\Xi_{pH} = (A, B_1, B_2, C)$ and $\tilde{\Xi}_{pH} = (\tilde{A}, \tilde{B}_1, \tilde{B}_2, \tilde{C})$ are feedback equivalent via a diffeomorphism $\tilde{x} = \phi(x)$ and an invertible feedback transformation $(w_1, w_2) = (\psi_1(x, \tilde{w}), \psi_2(x, \tilde{w}))$, then $w_1 = \psi_1(x, \tilde{w}) = \alpha_1(x) + \beta_1(x)\tilde{w}_1$ and $w_2 = \psi_2(x, \tilde{w}) = \alpha_2(x) + \beta_2(x)\tilde{w}_2$, where $\beta_i \neq 0$. Moreover,

$$\begin{aligned} \tilde{A} &= \phi_*(\beta_1\beta_2 A), & \tilde{B}_1 &= \phi_*(\alpha_2\beta_1 A + \beta_1 B_1), \\ \tilde{B}_2 &= \phi_*(\alpha_1\beta_2 A + \beta_2 B_2), & \tilde{C} &= \phi_*(C + \alpha_1\alpha_2 A + \alpha_1 B_1 + \alpha_2 B_2). \end{aligned} \tag{4.29}$$

(ii) Conversely, if a diffeomorphism $\tilde{x} = \phi(x)$ and a 4-tuple of functions $(\alpha_1, \alpha_2, \beta_1, \beta_2)$, with $\beta_i(\cdot) \neq 0$, satisfy (4.29), then the feedback transformation $\tilde{x} = \phi(x)$ and $\psi(x, \tilde{w}) = (\alpha_1 + \beta_1\tilde{w}_1, \alpha_2 + \beta_2\tilde{w}_2)$ brings Ξ_{pH} into $\tilde{\Xi}_{pH}$.

Remark (Locality of the results). When we introduced the definition of p-hyperbolic systems Ξ_{pH} we assumed that this form holds around an arbitrary point (x_0, w_0). We see that the pure feedback transformation $w = \psi(x, \tilde{w})$ that conjugate p-hyperbolic systems is global with respect to w. Therefore, in all results below, we will consider the form Ξ_{pH} *locally* around x_0 and *globally* in w. ◆

Proof.

(i) Clearly, diffeomorphisms of $\mathcal{X}$ map p-hyperbolic systems into p-hyperbolic systems and we have to show that only feedback transformations $w = \psi(x, \tilde{w})$ of the form $(\alpha_1(x)+\beta_1(x)\tilde{w}_1, \alpha_2(x)+\beta_2(x)\tilde{w}_2)$ conjugate p-hyperbolic systems. Applying $(w_1, w_2) = (\psi_1(x, \tilde{w}), \psi_2(x, \tilde{w}))$ to Ξ_{pH} yields,

$$\dot{x} = A\psi_1\psi_2 + B_2\psi_1 + B_2\psi_2 + C. \tag{4.30}$$

Since $A = A(x)$ and $B_i = B_i(x)$ are linearly independent, we see that, in order to preserve the p-hyperbolic structure of the system, the functions ψ_1, ψ_2, and $\psi_1\psi_2$ have to be of degree at most 1 in $\tilde{w}_i$. Hence,

$$\frac{\partial^2 \psi_i}{\partial \tilde{w}_j^2} = \frac{\partial^2 \psi_1 \psi_2}{\partial \tilde{w}_j^2} = 0, \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad j = 1, 2.$$

Up to a permutation of $\tilde{w}_1$ and $\tilde{w}_2$, solutions to those equations are given by, $\psi_1(x, \tilde{w}) = \alpha_1(x) + \beta_1(x)\tilde{w}_1$ and $\psi_2(x, \tilde{w}) = \alpha_2(x) + \beta_2(x)\tilde{w}_2$. Since the feedback is invertible with respect to $\tilde{w}$ we also have $\beta_1\beta_2 \neq 0$. Secondly, establishing relation (4.29) is a straighforward computation from (4.30) using $\psi = (\alpha_1 + \beta_1\tilde{w}_1, \alpha_2 + \beta_2\tilde{w}_2)$ and identifying second order and affine terms.

(ii) Conversely, for ϕ and $(\alpha_1, \alpha_2, \beta_2, \beta_2)$ satisfying (4.29), we clearly establish feedback equivalence of Ξ_{pH} and $\tilde{\Xi}_{pH}$ via $\tilde{x} = \phi(x)$ and $w = \psi(x, \tilde{w}) = (\alpha_1 + \beta_1\tilde{w}_1, \alpha_2 + \beta_2\tilde{w}_2)$.

■

We will develop relations involving structure functions attached to the 4-tuple (A, B_1, B_2, C) only and thus independent from diffeomorphisms of $\mathcal{X}$. So we will act on (A, B_1, B_2, C) by $(\alpha, \beta) = (\alpha_1, \alpha_2, \beta_1, \beta_2)$ and we will denote by $(\tilde{A}, \tilde{B}_1, \tilde{B}_2, \tilde{C})$ the result of that action (given by (4.29) with $\phi = \mathrm{Id}$), called a reparametrisation. For a p-hyperbolic system Ξ_{pH}, we call the triple (A, B_1, B_2) a *p-hyperbolic frame* (a pH-frame shortly) and we introduce the structure functions μ_i^k, and ν^k for $i = 1, 2$ and $k = 0, 1, 2$ defined by

$$[A, B_i] = \mu_i^0 A + \mu_i^1 B_1 + \mu_i^2 B_2, \quad \text{and} \quad [B_1, B_2] = \nu^0 A + \nu^1 B_1 + \nu^2 B_2.$$

Notice that we keep using symbols μ_i^k and ν^k to denote structure functions but they have nothing in common with structure functions of the previous section.

Definition 4.12 (Types of p-hyperbolic frames)**.** Consider a pH-frame (A, B_1, B_2), we define the following subclasses:

(a) *pseudo-commutative pH-frame* if $[A, B_i] = 0 \mod \operatorname{span}\{A\}$, that is $\mu_i^1 = \mu_i^2 = 0$ for $i = 1, 2$;

(b) *almost-commutative pH-frame* if $[A, B_i] = [B_1, B_2] = 0 \mod \operatorname{span}\{A\}$, that is $\mu_i^1 = \mu_i^2 = \nu^1 = \nu^2 = 0$ for $i = 1, 2$;

(c) *commutative pH-frame* if $[A, B_i] = [B_1, B_2] = 0$, that is $\mu_i^k = \nu^k = 0$ for $i = 1, 2$ and $k = 0, 1, 2$;

Clearly, each class is nested in the next one and observe from (4.29) that the distribution $\operatorname{span}\{A\}$ is uniquely attached to a p-hyperbolic system and thus pseudo- and almost-commutative pH-frames are well-defined. Moreover, since reparametrisations act on C by adding a linear combinations of A and B_i, it suggests to introduce the decomposition

$$C = \gamma^0 A + \gamma^1 B_1 + \gamma^2 B_2.$$

The following technical lemma shows how the three sets of structure functions μ_i^k, ν^k and γ^k are transformed under reparametrisations (α, β).

Lemma 4.2 (Structure functions transformations)**.** *Let $\Xi_{pH} = (A, B_1, B_2, C)$ and $\tilde{\Xi}_{pH} = (\tilde{A}, \tilde{B}_1, \tilde{B}_2, \tilde{C})$ be two feedback equivalent p-hyperbolic systems with structure functions μ_i^k, ν^k, γ^i, and $\tilde{\mu}_i^k$, $\tilde{\nu}^k$, $\tilde{\gamma}^k$, respectively. Then we have,*

$$
\begin{aligned}
\tilde{\mu}_1^0 &= \beta_1 \left[\mu_1^0 - \alpha_1\mu_1^1 - \alpha_2\mu_1^2 + \mathrm{L}_A(\alpha_2) - \alpha_1\mathrm{L}_A(\ln(\beta_1))\right.\\
&\quad \left.-\alpha_2\mathrm{L}_A(\ln(\beta_2) - \mathrm{L}_{B_1}(\ln(\beta_1\beta_2))\right],\\
\tilde{\mu}_2^0 &= \beta_2 \left[\mu_2^0 - \alpha_1\mu_2^1 - \alpha_2\mu_2^2 + \mathrm{L}_A(\alpha_1) - \alpha_1\mathrm{L}_A(\ln(\beta_1))\right.\\
&\quad \left.-\alpha_2\mathrm{L}_A(\ln(\beta_2)) - \mathrm{L}_{B_2}(\ln(\beta_1\beta_2))\right],\\
\tilde{\mu}_1^1 &= (\beta_1\beta_2\mu_1^1 + \beta_2\mathrm{L}_A(\beta_1)), \quad \tilde{\mu}_2^2 = (\beta_1\beta_2\mu_2^2 + \beta_1\mathrm{L}_A(\beta_2)),\\
\tilde{\mu}_1^2 &= (\beta_1)^2\mu_1^2, \quad \tilde{\mu}_2^1 = (\beta_2)^2\mu_2^1,
\end{aligned} \tag{4.31}
$$

$$
\begin{aligned}
\tilde{\nu}^0 &= \left[\nu^0 - \alpha_2\nu^1 - \alpha_1\nu^2 - \alpha_1\mu_1^0 + \alpha_2\mu_2^0 + (\alpha_1)^2\mu_1^2 - (\alpha_2)^2\mu_2^1\right]\\
&\quad + \left[\alpha_1\alpha_2(\mu_1^1 - \mu_2^2) + \alpha_2\mathrm{L}_A(\alpha_1) - \alpha_1\mathrm{L}_A(\alpha_2) + \mathrm{L}_{B_1}(\alpha_1) - \mathrm{L}_{B_2}(\alpha_2)\right],\\
\tilde{\nu}^1 &= \beta_2\left[\nu^1 + \alpha_2\mu_2^1 - \alpha_1\mu_1^1 - \alpha_1\mathrm{L}_A(\ln(\beta_1)) - \mathrm{L}_{B_2}(\ln(\beta_1))\right],\\
\tilde{\nu}^2 &= \beta_1\left[\nu^2 + \alpha_2\mu_2^2 - \alpha_1\mu_1^2 + \alpha_2\mathrm{L}_A(\ln(\beta_2)) + \mathrm{L}_{B_1}(\ln(\beta_2))\right],
\end{aligned} \tag{4.32}
$$

$$
\tilde{\gamma}^0 = \frac{1}{\beta_1\beta_2}\left(\gamma^0 - \alpha_2\gamma^1 - \alpha_1\gamma^2 - \alpha_1\alpha_2\right), \quad \gamma^i = \frac{1}{\beta_i}\left(\gamma^i + \alpha_i\right) \; i = 1, 2. \tag{4.33}
$$

Moreover, the following relations between the structure functions always hold:

$$
\begin{aligned}
\mathrm{L}_A(\nu^0) - \mathrm{L}_{B_1}(\mu_2^0) + \mathrm{L}_{B_2}(\mu_1^0) &= \nu^0(\mu_1^1 + \mu_2^2) - \nu^1\mu_1^0 - \nu^2\mu_2^0,\\
\mathrm{L}_A(\nu^1) - \mathrm{L}_{B_1}(\mu_2^1) + \mathrm{L}_{B_2}(\mu_1^1) &= \nu^1\mu_2^2 - \nu^2\mu_2^1 - \mu_2^0\mu_1^1 + \mu_1^0\mu_2^1,\\
\mathrm{L}_A(\nu^2) - \mathrm{L}_{B_1}(\mu_2^2) + \mathrm{L}_{B_2}(\mu_1^2) &= -\nu^1\mu_1^2 + \nu^2\mu_1^1 - \mu_2^0\mu_1^2 + \mu_1^0\mu_2^2.
\end{aligned} \tag{4.34}
$$

Proof. The computations are quite long and tedious so we leave them for Appendix 4.F. ■

In the following proposition we will characterise, via relations between the structure functions, the following normal form:

$$
\Xi'_{pH} \, : \, \dot{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} w_1 w_2 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} w_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} w_2 + C(x),
$$

which corresponds to the existence of a commutative pH-frame.

Proposition 4.9 (Existence of a commutative pH-frame)**.** *Consider a p-hyperbolic system Ξ_{pH} with pH-frame (A, B_1, B_2) and with structure functions μ_i^k for $i = 1, 2$ and $k = 0, 1, 2$. Then the following statements are equivalent:*

(i) *Ξ_{pH} is feedback equivalent to Ξ'_{pH},*

(ii) *There exists a feedback reparametrisation (α, β) such that $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$ is a commutative pH-frame.*

(iii) *There exists a feedback reparametrisation (α, β) such that $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$ is an almost-commutative pH-frame.*

(iv) *There exists a feedback reparametrisation (α, β) such that $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$ is a pseudo-commutative pH-frame.*

(v) *The structure functions μ_i^k satisfy*

$$
\mu_1^2 = \mu_2^1 = 0. \tag{4.35}
$$

A general pH-frame defines nine structure functions and it is striking that the normalisation of only two of them is sufficient for the normalisation of all. But it is somehow expected (that we have only two conditions) because we act on this set of nine structure functions with four arbitrary functions (α, β) and we always have three relations guaranteed by the Jacobi identity (4.34).

Moreover, observe that condition (4.35) implies

$$[A, B_1] = 0 \mod \operatorname{span}\{A, B_1\}, \quad \text{and} \quad [A, B_2] = 0 \mod \operatorname{span}\{A, B_2\},$$

and notice, by (4.29), that the distributions $\operatorname{span}\{A, B_i\}$ are invariant under reparametrisations. Compare this observation with the remark below Theorem 4.11 of the previous subsection.

Proof. We will show *(i)*⇒*(v)*⇒*(iv)*⇒*(iii)*⇒*(ii)*⇒*(i)*.

For *(i)*⇒*(v)*. The system Ξ'_{pH} satisfies $\tilde{\mu}_1^2 = \tilde{\mu}_2^1 = 0$ and by relation (4.31) we conclude that for Ξ_{pH} we necessarily have $\mu_1^2 = \mu_2^1 = 0$.

Conversely, for *(v)*⇒*(iv)*, assume $\mu_1^2 = \mu_2^1 = 0$ then from (4.31) we immediately have $\tilde{\mu}_1^2 = \tilde{\mu}_2^1 = 0$ for any reparametrisation $(w_1, w_2) = (\alpha_1 + \beta_1\tilde{w}_1, \alpha_2 + \beta_2\tilde{w}_2)$. We apply the reparametrisation given by $\alpha = 0$ and by any non trivial solutions of the equations $\mathrm{L}_A(\beta_i) = -\beta_i\mu_i^i$, for $i = 1, 2$ (no Einstein summation convention); to ensure that $\beta_i(\cdot) \neq 0$ we may actually solve $\mathrm{L}_A(\ln(\beta_i)) = -\mu_i^i$. Applying formula (4.31) to the transformed pH-frame $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$ we obtain $\tilde{\mu}_j^i = 0$ for $i, j = 1, 2$ and thus we have produced a pseudo-commutative pH-frame. Notice that a pseudo-commutative pH-frame is preserved by any reparametrisation satisfying $\mathrm{L}_{\tilde{A}}(\beta_i) = 0$.

(iv)⇒*(iii)*. Assume that $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$ is a pseudo-commutative frame with structure functions $(\tilde{\mu}_i^k, \tilde{\nu}^k)$. We claim that there exists smooth solutions of the following systems:

$$\begin{cases} \mathrm{L}_{\tilde{A}}(\ln\beta_1) &= 0 \\ \mathrm{L}_{\tilde{B}_2}(\ln\beta_1) &= \tilde{\nu}^1 \end{cases} \quad \text{and} \quad \begin{cases} \mathrm{L}_{\tilde{A}}(\ln\beta_2) &= 0 \\ \mathrm{L}_{\tilde{B}_1}(\ln\beta_2) &= -\tilde{\nu}^2 \end{cases}.$$

Indeed, on one hand the integrability conditions for the above systems read $\mathrm{L}_{\tilde{A}}(\tilde{\nu}^1) = 0$ and $\mathrm{L}_{\tilde{A}}(\tilde{\nu}^2) = 0$, respectively; on the other hand, the last two relations of (4.34), with $\tilde{\mu}_i^j = 0$, imply that we have $\mathrm{L}_{\tilde{A}}(\tilde{\nu}^i) = 0$. Therefore, integrability conditions are always fulfilled and thus smooth solutions of the above systems exist. Taking any solutions $\ln(\beta_i)$, we obtain a valid feedback transformation $(\tilde{w}_1, \tilde{w}_2) = (\beta_1\bar{w}_1, \beta_2\bar{w}_2)$ which, applied to $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$, construct a new pH-frame $(\bar{A}, \bar{B}_1, \bar{B}_2)$ satisfying $\bar{\nu}^i = 0$ and $\bar{\mu}_i^j = 0$, for $i, j = 1, 2$. Hence, $(\bar{A}, \bar{B}_1, \bar{B}_2)$ is an almost-commutative pH-frame.

(iii)⇒*(ii)*. Let $(\bar{A}, \bar{B}_1, \bar{B}_2)$ be an almost-commutative pH-frame, that is, $\bar{\mu}_i^j = \bar{\nu}^j = 0$ for $i, j = 1, 2$. Take any solutions α_i of the equations $\mathrm{L}_{\bar{A}}(\alpha_1) = -\bar{\mu}_2^0$ and $\mathrm{L}_{\bar{A}}(\alpha_2) = -\bar{\mu}_1^0$ and apply the reparametrisation $(\bar{w}_1, \bar{w}_2) = (\tilde{w}_1 + \alpha_1, \tilde{w}_2 + \alpha_2)$. Thus we obtain a new pH-frame $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$ satisfying $\left[\tilde{A}, \tilde{B}_i\right] = 0$, for $i = 1, 2$. Finally, take α_2 as a solution of the system

$$\begin{cases} \mathrm{L}_{\tilde{A}}(\alpha_2) &= 0 \\ \mathrm{L}_{\tilde{B}_2}(\alpha_2) &= \tilde{\nu}^0 \end{cases},$$

whose integrability condition $\mathrm{L}_{\tilde{A}}(\tilde{\nu}^0) = 0$ is guaranteed by the first equation of (4.34). Applying $(\tilde{w}_1, \tilde{w}_2) = (w_1, w_2 + \alpha_2)$, we obtain a new pH-frame satisfying $\mu_i^k = \nu^k = 0$, for $i = 1, 2$ and $k = 0, 1, 2$, that is a commutative pH-frame.

(ii)$\Rightarrow$*(i)*. Consider a p-hyperbolic system Ξ_{pH} such that (A, B_1, B_2) is a commutative pH-frame, apply a diffeomorphism $\tilde{x} = \phi(x)$ such that $\phi_* A = \frac{\partial}{\partial z}$, $\phi_* B_1 = \frac{\partial}{\partial y_1}$, and $\phi_* B_2 = \frac{\partial}{\partial y_2}$. In those coordinates, Ξ_{pH} takes the form Ξ'_{pH}. ■

Remark (Summary of the construction of a commutative pH-frame). Under relation (4.35), the proof *(v)*$\Rightarrow$*(ii)* of the above theorem consists in, successively, constructing a feedback (α, β) given by solutions of the following systems of equations

$$
\begin{aligned}
&\begin{cases} \mathrm{L}_A\left(\ln(\beta_1)\right) &= -\mu_1^1 \\ \mathrm{L}_{B_2}\left(\ln(\beta_1)\right) &= \nu^1 \end{cases}, \quad \begin{cases} \mathrm{L}_A\left(\ln(\beta_2)\right) &= -\mu_2^2 \\ \mathrm{L}_{B_1}\left(\ln(\beta_2)\right) &= -\nu^2 \end{cases}, \\
&\mathrm{L}_A\left(\alpha_1\right) = \mathrm{L}_{B_2}\left(\ln(\beta_2) - \mu_2^0 - \alpha_1\mu_1^1 + \nu^1, \quad \text{and,} \right.\\
&\begin{cases} \mathrm{L}_A\left(\alpha_2\right) &= \mathrm{L}_{B_1}\left(\ln(\beta_1) - \mu_1^0 - \alpha_2\mu_2^2 - \nu^2\right. \\ \mathrm{L}_{B_2}\left(\alpha_2\right) &= \alpha_2\mathrm{L}_{B_2}\left(\ln(\beta_2)\right) - \alpha_1\mathrm{L}_{B_1}\left(\ln(\beta_1)\right) + \mathrm{L}_{B_1}\left(\alpha_1\right) + \nu^0 \end{cases}
\end{aligned}
$$

obtained from equations (4.31) and (4.32) with $\tilde{\mu}_i^k = \tilde{\nu}^k = 0$, for $i = 1, 2$ and $k = 0, 1, 2$. The existence of solutions β of the first two systems requires two integrability conditions which are always granted by the last two equations of (4.34). Then, a solution α_1 of the middle equation always exists and, finally, a solution α_2 is guaranteed by the integrability condition given by the first relation of (4.34). ◆

In the remaining part of this section we will fully characterise the following two normal forms of p-hyperbolic systems:

$$
\Xi''_{pH} \; : \; \dot{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} w_1 w_2 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} w_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} w_2 + \begin{pmatrix} c_0(x) \\ 0 \\ 0 \end{pmatrix}
$$

$$
\Xi'''_{pH} \; : \; \dot{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} w_1 w_2 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} w_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} w_2 + \begin{pmatrix} c_0 \\ 0 \\ 0 \end{pmatrix}, \quad c_0 \in \mathbb{R}.
$$

Recall that p-hyperbolic systems Ξ_{pH} are parametrisation of p-hyperbolic submanifolds $\mathcal{S}_{pH}$ given by the triple $(\frac{\partial}{\partial z}, Q, b, c)$, with Q a symmetric matrix of constant signature $(1,1)$ acting on leaves of the distribution span$\left\{\frac{\partial}{\partial z}\right\}$, b a differential one-form in ann$\left(\frac{\partial}{\partial z}\right)$, and c is a smooth function. The first form Ξ''_{pH} is interesting because it represents a parametrisation of the following parabolic-hyperboloid submanifold

$$
\mathcal{S}''_{pH} = \left\{\dot{z} = \dot{y}_1\dot{y}_2 + c_0(x)\right\},
$$

that is, a submanifold with normalised matrix Q and normalised one-form b. The form Ξ'''_{pH} characterises the submanifolds without functional parameters, namely, those that do not depend on the point $x \in \mathcal{X}$. Moreover, in the latter case we will see that this normal form can be brought into a canonical form with either $c_0 = 0$ or $c_0 = 1$, see Proposition 4.10.

Our conditions will be expressed for Ξ_{pH} in terms of structure functions, and therefore are checkable on any p-hyperbolic system Ξ_{pH}. However, those conditions will be quite complicated to analyse, hence it will be convenient to give, as a corollary, the same conditions for the system Ξ'_{pH}, that is in a commutative pH-frame. We define the function

$$
\Gamma = \gamma^0 + \gamma^1\gamma^2
$$

that by diffeomorphisms ϕ of $\mathcal{X}$ is transformed by $\tilde{\Gamma}\circ\phi = \Gamma$ and under reparametrisations (α,β) by $\beta_1\beta_2\tilde{\Gamma} = \Gamma$, as it can be computed from (4.33).

Theorem 4.13 (Classification results of p-hyperbolic systems)**.** *Consider a p-hyperbolic system $\Xi_{pH} = (A, B_1, B_2, C)$ with structure functions $(\mu_i^k, \nu^k, \gamma^k)$. We have*

(i) Ξ_{pH} is equivalent to Ξ'_{pH} if and only if

$$\mu_1^2 = \mu_2^1 = 0. \tag{4.35}$$

(ii) Ξ_{pH} is equivalent to Ξ''_{pH} if and only if (4.35) holds and, additionally, we have

$$\begin{aligned}\gamma^1 \mathrm{L}_A\left(\gamma^2\right) - \gamma^2 \mathrm{L}_A\left(\gamma^1\right) + \mathrm{L}_{B_1}\left(\gamma^1\right) - \mathrm{L}_{B_2}\left(\gamma^2\right) = \\ \nu^0 + \gamma^1\left(\mu_1^0 - \gamma^2\mu_2^2 + \nu^2\right) + \gamma^2\left(-\mu_2^0 + \gamma^1\mu_1^1 + \nu^1\right),\end{aligned} \tag{4.36}$$

$$\begin{aligned}\mathrm{L}_A^2\left(\gamma^1\right) &= \mathrm{L}_A\left(\mu_2^0 - \gamma^1\mu_1^1 - \nu^1\right) + \mathrm{L}_{B_2}\left(\mu_2^2\right) \\ &\qquad + \mu_2^2\left(\gamma^1\mu_1^1 + \nu^1 + \mathrm{L}_A\left(\gamma^1\right)\right), \\ \mathrm{L}_A^2\left(\gamma^2\right) &= \mathrm{L}_A\left(\mu_1^0 - \gamma^2\mu_2^2 + \nu^2\right) + \mathrm{L}_{B_1}\left(\mu_1^1\right) \\ &\qquad - \mu_1^1\left(-\gamma^2\mu_2^2 + \nu^2 - \mathrm{L}_A\left(\gamma^2\right)\right),\end{aligned} \tag{4.37}$$

$$\begin{aligned}\mathrm{L}_{B_1}\left(\mathrm{L}_A\left(\gamma^1\right)\right) &= \mathrm{L}_{B_2}\left(\nu^2\right) + \mathrm{L}_{B_1}\left(\mu_2^0 - \gamma^1\mu_1^1 - \nu^1\right) + \mu_2^2\nu^0 \\ &\qquad + 2\nu^1\nu^2 - \nu^2\left(\mu_2^0 - \gamma^1\mu_1^1 - \mathrm{L}_A\left(\gamma^1\right)\right), \\ \mathrm{L}_{B_2}\left(\mathrm{L}_A\left(\gamma^2\right)\right) &= -\mathrm{L}_{B_1}\left(\nu^1\right) + \mathrm{L}_{B_2}\left(\mu_1^0 - \gamma^2\mu_2^2 + \nu^2\right) - \mu_1^1\nu^0 \\ &\qquad + 2\nu^1\nu^2 + \nu^1\left(\mu_1^0 - \gamma^2\mu_2^2 - \mathrm{L}_A\left(\gamma^2\right)\right).\end{aligned} \tag{4.38}$$

(iii) Ξ_{pH} is equivalent to Ξ'''_{pH} if and only if (4.35), (4.36), (4.37), (4.38) hold and, additionally, we have

$$\begin{aligned}\mathrm{L}_A\left(\Gamma\right) + \Gamma(\mu_1^1 + \mu_2^2) &= 0, \\ \mathrm{L}_{B_1}\left(\Gamma\right) + \Gamma\mathrm{L}_A\left(\gamma^2\right) - \Gamma\left(\mu_1^0 - \gamma^2\mu_2^2\right) &= 0, \\ \mathrm{L}_{B_2}\left(\Gamma\right) + \Gamma\mathrm{L}_A\left(\gamma^1\right) - \Gamma\left(\mu_2^0 - \gamma^1\mu_1^1\right) &= 0.\end{aligned} \tag{4.39}$$

Remark (Idea of the theorem). The idea behind statement *(ii)* of the above proposition is the following. For Ξ''_{pH}, with structure functions $(\tilde{\mu}_i^k, \tilde{\nu}^k, \tilde{\gamma}^k)$, we have $\tilde{\mu}_i^k = \tilde{\nu}^k = 0$ (i.e. a commutative pH-frame exists) and $\tilde{\gamma}^1 = \tilde{\gamma}^2 = 0$. Relation (4.33) imposes that we have $\alpha_1 = -\gamma^1$ and $\alpha_2 = -\gamma^2$, thus α is fixed and the group of reprametrisations now depends on an arbitrary β only. Conditions (4.35), (4.36), (4.37), and (4.38) describe then the existence of a reparametrisation β (we do not change α_1 and α_2) such that a commutative pH-frame exists. The construction of this reparametrisation is given by solutions of two systems of first order partial differential equations (see the proof below) and thus some integrability conditions are required. Those integrability conditions are given by (4.37) and (4.38); notice that only 2 integrability conditions are required by each systems (instead of the expected 3) because one is always fulfilled by one of the last two relations of (4.34). And condition (4.36) ensures that for the new frame $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$ we have $\tilde{\nu}^0 = 0$.
The idea behind statement *(iii)* is mostly the same, the additional condition (4.39) ensures that the resulting function c_0 of Ξ''_{pH} is constant. Indeed for system Ξ''_{pH}, we have $\Gamma = c_0$ and relation (4.39) implies that $\frac{\partial c_0}{\partial z} = \frac{\partial c_0}{\partial y_1} = \frac{\partial c_0}{\partial y_2} = 0$, i.e. c_0 is constant. ◆

Proof.

(i) It is Proposition 4.9.

(ii) Assume that Ξ_{pH}, with structure functions $(\mu_i^k, \nu^k, \gamma^k)$, is equivalent to Ξ''_{pH}, with structure functions $\tilde{\mu}_i^k = \tilde{\nu}^k = 0$, $\tilde{\gamma}^1 = \tilde{\gamma}^2 = 0$, and $\tilde{\gamma}^0 = c_0$. The necessity of (4.35) is immediate by Proposition 4.9. Using relation (4.33) with $\tilde{\gamma}^1 = \tilde{\gamma}^2 = 0$ we obtain that $\alpha_1 = -\gamma^1$ and $\alpha_2 = -\gamma^2$. Moreover, from (4.31) and (4.32) with $\tilde{\mu}_i^k = \tilde{\nu}^k = 0$ we deduce, first (with $\tilde{\nu}^0 = 0$) the necessity of (4.36) and, second, the following systems of first order partial differential equations for $\ln(\beta_i)$:

$$\begin{cases} \mathrm{L}_A\left(\ln(\beta_1)\right) &= -\mu_1^1 \\ \mathrm{L}_{B_1}\left(\ln(\beta_1)\right) &= -\mathrm{L}_A\left(\gamma^2\right) + \mu_1^0 - \gamma^2\mu_2^2 + \nu^2 \\ \mathrm{L}_{B_2}\left(\ln(\beta_1)\right) &= \nu^1 \end{cases},$$

$$\text{and} \quad \begin{cases} \mathrm{L}_A\left(\ln(\beta_2)\right) &= -\mu_2^2 \\ \mathrm{L}_{B_1}\left(\ln(\beta_2)\right) &= -\nu^2 \\ \mathrm{L}_{B_2}\left(\ln(\beta_2)\right) &= -\mathrm{L}_A\left(\gamma^1\right) + \mu_2^0 - \gamma^1\mu_1^1 - \nu^1 \end{cases}.$$

Those two systems imply 6 integrability conditions which are necessary, but two of them are always given by the last two equations of (4.34) and the other four are given by (4.37) and (4.38).

Conversely, assume that (4.35), (4.36), (4.37), and (4.38) hold. Then there exists solutions $\ln(\beta_i)$ of the above systems, and applying to Ξ_{pH} the reparametrisation $(w_1, w_2) = (\beta_1\tilde{w}_1 - \gamma^1, \beta_2\tilde{w}_2 - \gamma^2)$ yields the system Ξ''_{pH}.

(iii) Assume that Ξ_{pH}, with structure functions $(\mu_i^k, \nu^k, \gamma^k)$, is equivalent to Ξ'''_{pH}, with structure functions $\tilde{\mu}_i^k = \tilde{\nu}^k = 0$, $\tilde{\gamma}^1 = \tilde{\gamma}^2 = 0$, and $\tilde{\gamma}^0 = c_0 \in \mathbb{R}$. The necessity of (4.35), (4.36), (4.37), (4.38) is clear from the previous item of the proof and we show that (4.39) is necessary. For Ξ'''_{pH}, we have $\tilde{\Gamma} = c_0 \in \mathbb{R}$, and under a reparametrisation we have $\tilde{\Gamma} = \frac{\Gamma}{\beta_1\beta_2}$, where $\Gamma = \gamma^0 + \gamma^1\gamma^2$. Differentiating the last equation along $\tilde{A}$, $\tilde{B}_1$, and $\tilde{B}_2$, yields relation (4.39).

Conversely, assume that Ξ_{pH} satisfy (4.35), (4.36), (4.37), (4.38), and (4.39). Then by statement *(ii)*, the system Ξ_{pH} can be brought into form Ξ''_{pH} for which we have $(A, B_1, B_2) = \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right)$ and $\Gamma = c_0(x)$, thus (4.39) reads $\frac{\partial c_0}{\partial z} = \frac{\partial c_0}{\partial y_1} = \frac{\partial c_0}{\partial y_2} = 0$ and finally $c_0 \in \mathbb{R}$, i.e. we, actually, have the normal form Ξ'''_{pH}.

∎

As announced we represent the conditions of the previous theorem in a commutative pH-frame in order to give an interpretation of them.

Corollary 4.3 (Classification of Ξ'_{pH})**.** *Consider a p-hyperbolic system* $\Xi'_{pH} = (A, B_1, B_2, C)$ *with structure functions* $(\mu_i^k, \nu^k, \gamma^k) = (0, 0, \gamma^k)$.

(i) Ξ'_{pH} *is equivalent to* Ξ''_{pH} *if and only if it holds*

$$\gamma^1\mathrm{L}_A\left(\gamma^2\right) - \gamma^2\mathrm{L}_A\left(\gamma^1\right) + \mathrm{L}_{B_1}\left(\gamma^1\right) - \mathrm{L}_{B_2}\left(\gamma^2\right) = 0, \tag{4.36'}$$

$$\mathrm{L}_A^2\left(\gamma^1\right) = \mathrm{L}_A^2\left(\gamma^2\right) = 0, \tag{4.37'}$$

$$\mathrm{L}_{B_1}\left(\mathrm{L}_A\left(\gamma^1\right)\right) = \mathrm{L}_{B_2}\left(\mathrm{L}_A\left(\gamma^2\right)\right) = 0. \tag{4.38'}$$

(ii) Ξ'_{pH} *is equivalent to* Ξ'''_{pH} *if and only if (4.36'), (4.37'), (4.38') hold, and, additionally, we have*

$$\text{(4.39')} \qquad \mathrm{L}_A(\Gamma) = \mathrm{L}_{B_1}(\Gamma) + \Gamma \mathrm{L}_A\left(\gamma^2\right) = \mathrm{L}_{B_2}(\Gamma) + \Gamma \mathrm{L}_A\left(\gamma^1\right) = 0.$$

Remark (Interpretation of the conditions). Consider the system Ξ'_{pH} with a commutative pH-frame (A, B_1, B_2) and with structure functions $\mu_i^k = \nu^k = 0$ and γ^k. Conditions (4.36'), (4.37'), and (4.38') translate the fact that there exists a reparametrisation (α, β) that, both, preserves the commutativity of the pH-frame and ensures that we obtain $\tilde{\gamma}^1 = \tilde{\gamma}^2 = 0$. In the rectified frame $(A, B_1, B_2) = \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right)$, the meaning of conditions (4.37') and (4.38') is clear: they imply that we have

$$\text{(4.40)} \qquad \gamma^1 = \gamma_1^1(y_2)z + \gamma_2^1(y), \quad \text{and} \quad \gamma^2 = \gamma_1^2(y_1)z + \gamma_2^2(y).$$

Then, condition (4.36') gives a relation between the functions γ_j^i whose interpretation is not so clear yet. Therefore, the systems Ξ'_{pH} that are equivalent to Ξ''_{pH} are parametrised by one arbitrary smooth function of 3 variables, namely $\gamma^0(x)$, and two smooth functions γ^1 and γ^2 of the form (4.40) and satisfying (4.36').

Assume that Ξ'_{pH} additionally satisfies (4.39'). Clearly, the smooth solutions Γ of (4.39') are given by

$$\Gamma(y) = G \exp\left(\int -\gamma_1^1 \,\mathrm{d}y_2\right) \exp\left(\int -\gamma_1^2 \,\mathrm{d}y_1\right), \quad G \in \mathbb{R}.$$

Therefore the systems Ξ'_{pH} that are equivalent to Ξ'''_{pH} are parametrised by a real constant and two smooth functions γ^1 and γ^2 of the form (4.40) and satisfying (4.36'). If that constant satisfies $G = 0$ then Ξ'_{pH} is equivalent to Ξ^0_{pH} otherwise if $G \neq 0$ then Ξ'_{pH} is equivalent to Ξ^1_{pH}, see proposition below. ◆

The following proposition gives a canonical form of systems Ξ'''_{pH} depending on whether $c_0 = 0$ or $c_0 \neq 0$.

Proposition 4.10 (Canonical form of Ξ'''_{PH})**.** *Consider a p-hyperbolic system* Ξ_{pH} *with structure functions* $(\mu_i^k, \nu^k, \gamma^k)$ *satisfying (4.35), (4.36), (4.37), (4.38), and (4.39). Then, it always admits one of the following canonical form*

$$\Xi^0_{pH} \,:\, \dot{x} = \begin{pmatrix}1\\0\\0\end{pmatrix} w_1 w_2 + \begin{pmatrix}0\\1\\0\end{pmatrix} w_1 + \begin{pmatrix}0\\0\\1\end{pmatrix} w_2 + \begin{pmatrix}0\\0\\0\end{pmatrix}, \quad \textit{or}$$

$$\Xi^1_{pH} \,:\, \dot{x} = \begin{pmatrix}1\\0\\0\end{pmatrix} w_1 w_2 + \begin{pmatrix}0\\1\\0\end{pmatrix} w_1 + \begin{pmatrix}0\\0\\1\end{pmatrix} w_2 + \begin{pmatrix}1\\0\\0\end{pmatrix}.$$

Moreover, Ξ_{pH} *is equivalent to the former if and only if* $\Gamma \equiv 0$, *and to the latter if and only if* $\Gamma \neq 0$.

Notice that if Ξ_{pH} is feedback equivalent to Ξ^0_{pH} and thus $\Gamma \equiv 0$, then condition (4.39) is automatically satisfied.

Proof. If Ξ_{pH} satisfies (4.35), (4.36), (4.37), (4.38), and (4.39), then it is equivalent to Ξ'''_{pH}. If $c_0 = 0$, then we have the canonical form Ξ^0_{pH}, otherwise use the following transformation $(w_1, w_2) = (c_0\tilde{w}_1, \tilde{w}_2)$ and $(\tilde{z}, \tilde{y}_1, \tilde{y}_2) = \left(\frac{z}{c_0}, \frac{y_1}{c_0}, y_2\right)$ to obtain Ξ^1_{pH}.

Since we always have $\tilde{\Gamma} = \frac{\Gamma}{\beta_1\beta_2}$, clearly Ξ_{pH} (satisfying (4.35), (4.36), (4.37), (4.38), and (4.39)) is equivalent to Ξ^0_{pH} if and only if $\Gamma \equiv 0$ and is equivalent to Ξ^1_{pH} if and only if $\Gamma \neq 0$. ∎

We finish this subsection by explaining how to get normal and canonical forms of p-hyperbolic submanifolds $\mathcal{S}_{pH}$. Recall that those submanifolds are given by the equation $\dot{z} = \dot{y}^t Q(x)\dot{y} + b_1(x)\dot{y}_1 + b_2\dot{y}_2 + c(x)$, with $Q = \left(\begin{smallmatrix} q_1 & q_2 \\ q_2 & q_3 \end{smallmatrix}\right)$ satisfying $\Delta_2 := \det(Q) = q_1q_3 - (q_2)^2 < 0$. Hence a direct parametrisation of $\mathcal{S}_{pH}$ is given by

$$\Xi_{\mathcal{S}_{pH}} : \begin{cases} \dot{z} &= q_1(w_1)^2 + 2q_2w_1w_2 + q_3(w_2)^2 + b_1w_1 + b_2w_2 + c \\ \dot{y}_1 &= w_1 \\ \dot{y}_2 &= w_2 \end{cases},$$

where all functions q_1, q_2, q_3, b_1, b_2, and c depend on x. Moreover, observe that we have either $q_2 + \sqrt{-\Delta_2} \neq 0$ or $q_2 - \sqrt{-\Delta_2} \neq 0$, thus one of the reparametrisation

$$\begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} = \begin{pmatrix} \frac{q_1}{q_2 \pm \sqrt{-\Delta_2}} & 1 \\ q_2 \pm \sqrt{-\Delta_2} & q_3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

transforms $\Xi_{\mathcal{S}_{pH}}$ into a system $\Xi^{\mathcal{S}}_{pH}$ (which is of the form Ξ_{pH} and the upper script $\mathcal{S}$ indicates that it is a parametrisation of the submanifold $\mathcal{S}_{pH}$) satisfying $A = \frac{\partial}{\partial z}$ with the fields B_i depending on the functions q_i and b_i, and $C = c(x)\frac{\partial}{\partial z}$. The conditions of the previous results can be tested on $\Xi^{\mathcal{S}}_{pH}$ and the normal forms obtained above give normal forms of $\mathcal{S}_{pH}$. Precisely, we have

Corollary 4.4 (Normal and canonical forms of p-hyperbolic submanifolds)**.** *Consider a p-hyperbolic submanifolds $\mathcal{S}_{pH} = \{\dot{z} = \dot{y}^t Q(x)\dot{y} + b_1(x)\dot{y}_1 + b_2(x)\dot{y}_2 + c(x)\}$, together with its parametrisation $\Xi^{\mathcal{S}}_{pH}$. The following statements hold:*

(i) If $\Xi^{\mathcal{S}}_{pH}$ is equivalent to Ξ'_{pH}, then $\mathcal{S}_{pH}$ is equivalent to $\mathcal{S}'_{pH} = \{\dot{z} = \dot{y}_1\dot{y}_2 + b_1(x)\dot{y}_1 + b_2(x)\dot{y}_2 + c(x)\}$.

(ii) If $\Xi^{\mathcal{S}}_{pH}$ is equivalent to Ξ''_{pH}, then $\mathcal{S}_{pH}$ is equivalent to $\mathcal{S}''_{pH} = \{\dot{z} = \dot{y}_1\dot{y}_2 + c(x)\}$.

(iii) If $\Xi^{\mathcal{S}}_{pH}$ is equivalent to Ξ'''_{pH}, then $\mathcal{S}_{pH}$ is equivalent to $\mathcal{S}'''_{pH} = \{\dot{z} = \dot{y}_1\dot{y}_2 + c\}$, with $c \in \mathbb{R}$; moreover, c can always be normalised to either $c = 0$ or $c = 1$.

The normal form S'_{pH} describes a normalisation of the matrix $Q(x)$, then the normal form $\mathcal{S}''_{pH}$ describes, additionally, a normalisation of the functions b_1 and b_2, and finally, the normal form $\mathcal{S}'''_{pH}$ describes the p-hyperbolic submanifolds with no functional parameters, i.e. those that do not depend on the point $x \in \mathcal{X}$.

In this subsection, we studied the classification problem of nonlinear p-hyperbolic systems under the group of feedback action. Our classification includes several normal and canonical forms. The conditions that we introduced are checkable in terms of algebraic and differential relations of structure functions attached to vector fields of p-hyperbolic systems.

3 Conclusion and Perspectives

In this chapter, we extended to the dimension $n = \dim(\mathcal{X}) = 3$ the results of the previous chapter. We studied the equivalence of submanifolds of the tangent bundle $T\mathcal{X}$ of $\mathcal{X}$ to quadric submanifolds, we particularly studied the case of paraboloid elliptic and paraboloid hyperbolic systems. We provided a complete characterisation of those submanifold via the study of the feedback equivalence of control-affine system (on a 5-dimensional manifold with two controls) to control-affine systems parametrising the paraboloid submanifolds. Then by working within the class of parametrisation of paraboloid submanifolds (seen as control-nonlinear systems on $\mathcal{X}$ with two controls), we studied the problem of equivalence of paraboloid submanifolds.

In the next chapter, we will extend all results of this chapter to the case of an arbitrary dimension $n = \dim(\mathcal{X})$. These generalisations will give a new insight and new interpretations of the results of this chapter.

The work done for this chapter left some very interesting problems that we plan to address later. First, it remains to give a characterisation and a classification of the three remaining non-degenerated quadrics of $T\mathcal{X}$, namely the ellipsoid and the one- or two-sheeted hyperboloid. A parametrisation as a control-nonlinear system of those submanifolds is given, respectively, by

$$\dot{x} = A(x)\cos(w_1) + B_1(x)\sin(w_1)\cos(w_2) + B_2(x)\sin(w_1)\sin(w_2) + C(x), \quad \text{or}$$
$$\dot{x} = A(x)\cosh(w_1) + B_1(x)\sinh(w_1)\cos(w_2) + B_2(x)\sinh(w_1)\sin(w_2) + C(x).$$

It will be very interesting to characterise each class separately but also to characterise the 5 quadrics together (as we did in the previous chapter for the case $n = 2$) and so we should be able to see if we can smoothly pass from one type to another.

Further in the future, we would like to study a characterisation of several parametrised surfaces, for instance one could be interested in a parametrisation of a general surface of revolution, or of a ruled surface etc, those parametrisation can be e.g.

$$\dot{x} = A(x)g(w_1) + B_1(x)f(w_1)\cos(w_2) + B_2(x)f(w_1)\sin(w_2) + C(x),$$
$$\dot{x} = F_1(x, w_1) + w_2 F_2(x, w_1).$$

The latter case is of special importance for studying the problem of dynamic feedback linearisation (see [Rou98]).

4.A Transformation of the structure functions attached to WOF

We show the following formula (4.3):

$$\begin{pmatrix} \mu^1 & \mu^2 \\ \mu^1_{1,2} & \mu^2_{1,2} \\ \mu^1_{2,1} & \mu^2_{2,1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda}\mathrm{L}_{g_1}(\lambda) + \mathrm{L}_{g_2}(\theta) & \mathrm{L}_{g_1}(\theta) - \frac{1}{\lambda}\mathrm{L}_{g_2}(\lambda) \\ -\mathrm{L}_{g_1}(\theta) & \frac{1}{\lambda}\mathrm{L}_{g_1}(\lambda) \\ \frac{1}{\lambda}\mathrm{L}_{g_2}(\lambda) & \mathrm{L}_{g_2}(\theta) \end{pmatrix}$$
$$+ \begin{pmatrix} \cos(2\theta) & \sin(2\theta) & \sin(2\theta) \\ -\frac{1}{2}\sin(2\theta) & \cos(\theta)^2 & -\sin(\theta)^2 \\ -\frac{1}{2}\sin(2\theta) & -\sin(\theta)^2 & \cos(\theta)^2 \end{pmatrix} \begin{pmatrix} \tilde{\mu}^1 & \tilde{\mu}^2 \\ \tilde{\mu}^1_{1,2} & \tilde{\mu}^2_{1,2} \\ \tilde{\mu}^1_{2,1} & \tilde{\mu}^2_{2,1} \end{pmatrix} \beta(\lambda,\theta).$$

Let $(\tilde{g}_1, \tilde{g}_2)$ be a WOF with its structure functions $(\tilde{\mu}^k)$ and $(\tilde{\mu}^k_{i,j})$ and take $(g_1, g_2) = (\tilde{g}_1, \tilde{g}_2)\beta(\lambda, \theta)$. Thus, we have

$$\begin{aligned} \mathrm{ad}_f g_1 &= \lambda\left(\cos(\theta)\mathrm{ad}_f\tilde{g}_1 + \sin(\theta)\mathrm{ad}_f\tilde{g}_2\right) \mod \mathcal{D}^0, \\ \mathrm{ad}_f g_2 &= \lambda\left(-\sin(\theta)\mathrm{ad}_f\tilde{g}_1 + \cos(\theta)\mathrm{ad}_f\tilde{g}_2\right) \mod \mathcal{D}^0. \end{aligned}$$

First, we have (every equality is considered modulo $\mathcal{D}^0$)

$$\begin{aligned} [g_1, \mathrm{ad}_f g_1] - [g_2, \mathrm{ad}_f g_2] &= \left[\lambda\left(\tilde{g}_1\cos(\theta) + \tilde{g}_2\sin(\theta)\right), \lambda\left(\cos(\theta)\mathrm{ad}_f\tilde{g}_1 + \sin(\theta)\mathrm{ad}_f\tilde{g}_2\right)\right] \\ &\quad - \left[\lambda\left(-\tilde{g}_1\sin(\theta) + \tilde{g}_2\cos(\theta)\right), \lambda\left(-\sin(\theta)\mathrm{ad}_f\tilde{g}_1 + \cos(\theta)\mathrm{ad}_f\tilde{g}_2\right)\right], \\ &= \lambda^2\left[\tilde{g}_1\cos(\theta) + \tilde{g}_2\sin(\theta), \cos(\theta)\mathrm{ad}_f\tilde{g}_1 + \sin(\theta)\mathrm{ad}_f\tilde{g}_2\right] + \frac{1}{\lambda}\mathrm{L}_{g_1}(\lambda)\,\mathrm{ad}_f g_1 \\ &\quad - \lambda^2\left[-\tilde{g}_1\sin(\theta) + \tilde{g}_2\cos(\theta), -\sin(\theta)\mathrm{ad}_f\tilde{g}_1 + \cos(\theta)\mathrm{ad}_f\tilde{g}_2\right] - \frac{1}{\lambda}\mathrm{L}_{g_2}(\lambda)\,\mathrm{ad}_f g_2, \\ &= \lambda\mathrm{L}_{g_1}(\cos(\theta))\,\mathrm{ad}_f\tilde{g}_1 + \lambda\mathrm{L}_{g_1}(\sin(\theta))\,\mathrm{ad}_f\tilde{g}_2 + \frac{1}{\lambda}\mathrm{L}_{g_1}(\lambda)\,\mathrm{ad}_f g_1 \\ &\quad - \lambda\mathrm{L}_{g_2}(-\sin(\theta))\,\mathrm{ad}_f\tilde{g}_1 - \lambda\mathrm{L}_{g_2}(\cos(\theta))\,\mathrm{ad}_f\tilde{g}_2 - \frac{1}{\lambda}\mathrm{L}_{g_2}(\lambda)\,\mathrm{ad}_f g_2 \\ &\quad + \lambda^2\left(\cos(\theta)^2[\tilde{g}_1, \mathrm{ad}_f\tilde{g}_1] + \cos(\theta)\sin(\theta)[\tilde{g}_1, \mathrm{ad}_f\tilde{g}_2]\right) \\ &\quad + \lambda^2\left(\sin(\theta)\cos(\theta)[\tilde{g}_2, \mathrm{ad}_f\tilde{g}_1] + \sin(\theta)^2[\tilde{g}_2, \mathrm{ad}_f\tilde{g}_2]\right) \\ &\quad - \lambda^2\left(\sin(\theta)^2[\tilde{g}_1, \mathrm{ad}_f\tilde{g}_1] - \cos(\theta)\sin(\theta)[\tilde{g}_1, \mathrm{ad}_f\tilde{g}_2]\right) \\ &\quad - \lambda^2\left(-\cos(\theta)\sin(\theta)[\tilde{g}_2, \mathrm{ad}_f\tilde{g}_1] + \cos(\theta)^2[\tilde{g}_2, \mathrm{ad}_f\tilde{g}_2]\right), \\ &= \lambda\mathrm{L}_{g_1}(\theta)\left(-\sin(\theta)\mathrm{ad}_f\tilde{g}_1 + \cos(\theta)\mathrm{ad}_f\tilde{g}_2\right) + \frac{1}{\lambda}\mathrm{L}_{g_1}(\lambda)\,\mathrm{ad}_f g_1 \\ &\quad - \lambda\mathrm{L}_{g_2}(\theta)\left(-\cos(\theta)\mathrm{ad}_f\tilde{g}_1 - \sin(\theta)\mathrm{ad}_f\tilde{g}_2\right) - \frac{1}{\lambda}\mathrm{L}_{g_2}(\lambda)\,\mathrm{ad}_f g_2 \\ &\quad + \lambda^2\left(\cos(\theta)^2 - \sin(\theta)^2\right)\left([\tilde{g}_1, \mathrm{ad}_f\tilde{g}_1] - [\tilde{g}_2, \mathrm{ad}_f\tilde{g}_2]\right) \\ &\quad + 2\lambda^2\cos(\theta)\sin(\theta)\left([\tilde{g}_1, \mathrm{ad}_f\tilde{g}_2] + [\tilde{g}_2, \mathrm{ad}_f\tilde{g}_1]\right), \\ &= \left(\frac{1}{\lambda}\mathrm{L}_{g_1}(\lambda) + \mathrm{L}_{g_2}(\theta)\right)\mathrm{ad}_f g_1 + \left(\mathrm{L}_{g_1}(\theta) - \frac{1}{\lambda}\mathrm{L}_{g_2}(\lambda)\right)\mathrm{ad}_f g_2 \\ &\quad + \lambda^2\left[\cos(2\theta)\tilde{\mu}^1 + \sin(2\theta)(\tilde{\mu}^1_{1,2} + \tilde{\mu}^1_{2,1})\right]\mathrm{ad}_f\tilde{g}_1 \\ &\quad + \lambda^2\left[\cos(2\theta)\tilde{\mu}^2 + \sin(2\theta)(\tilde{\mu}^2_{1,2} + \tilde{\mu}^2_{2,1})\right]\mathrm{ad}_f\tilde{g}_2. \end{aligned}$$

Second, we have

$$
\begin{aligned}
[g_1, \mathrm{ad}_f g_2] &= \left[\lambda\left(\tilde{g}_1 \cos(\theta) + \tilde{g}_2 \sin(\theta)\right), \lambda\left(-\sin(\theta)\mathrm{ad}_f\tilde{g}_1 + \cos(\theta)\mathrm{ad}_f\tilde{g}_2\right)\right], \\
&= \lambda^2 \left[\tilde{g}_1 \cos(\theta) + \tilde{g}_2 \sin(\theta), -\mathrm{ad}_f\tilde{g}_1 \sin(\theta) + \mathrm{ad}_f\tilde{g}_2 \cos(\theta)\right] + \frac{1}{\lambda}\mathrm{L}_{g_1}(\lambda)\,\mathrm{ad}_f g_2, \\
&= -\lambda \mathrm{L}_{g_1}(\sin(\theta))\,\mathrm{ad}_f\tilde{g}_1 + \lambda \mathrm{L}_{g_1}(\cos(\theta))\,\mathrm{ad}_f\tilde{g}_2 + \frac{1}{\lambda}\mathrm{L}_{g_1}(\lambda)\,\mathrm{ad}_f g_2 \\
&+ \lambda^2 \big(-\cos(\theta)\sin(\theta)\left[\tilde{g}_1, \mathrm{ad}_f\tilde{g}_1\right] + \cos(\theta)^2\left[\tilde{g}_1, \mathrm{ad}_f\tilde{g}_2\right]\big) \\
&+ \lambda^2 \big(-\sin(\theta)^2\left[\tilde{g}_2, \mathrm{ad}_f\tilde{g}_1\right] + \cos(\theta)\sin(\theta)\left[\tilde{g}_2, \mathrm{ad}_f\tilde{g}_2\right]\big), \\
&= \lambda \mathrm{L}_{g_1}(\theta)\left(-\cos(\theta)\mathrm{ad}_f\tilde{g}_1 - \sin(\theta)\mathrm{ad}_f\tilde{g}_2\right) + \frac{1}{\lambda}\mathrm{L}_{g_1}(\lambda)\,\mathrm{ad}_f g_2 \\
&+ \lambda^2 \left(-\cos(\theta)\sin(\theta)\left(\left[\tilde{g}_1, \mathrm{ad}_f\tilde{g}_1\right] - \left[\tilde{g}_2, \mathrm{ad}_f\tilde{g}_2\right]\right)\right) \\
&+ \lambda^2 \cos(\theta)^2\left[\tilde{g}_1, \mathrm{ad}_f\tilde{g}_2\right] - \lambda^2\sin(\theta)^2\left[\tilde{g}_2, \mathrm{ad}_f\tilde{g}_1\right], \\
&= -\mathrm{L}_{g_1}(\theta)\,\mathrm{ad}_f g_1 + \frac{1}{\lambda}\mathrm{L}_{g_1}(\lambda)\,\mathrm{ad}_f g_2 \\
&+ \lambda^2\left[-\frac{1}{2}\sin(2\theta)\tilde{\mu}^1 + \cos(\theta)^2\tilde{\mu}^1_{1,2} - \sin(\theta)^2\tilde{\mu}^1_{2,1}\right]\mathrm{ad}_f\tilde{g}_1 \\
&+ \lambda^2\left[-\frac{1}{2}\sin(2\theta)\tilde{\mu}^2 + \cos(\theta)^2\tilde{\mu}^2_{1,2} - \sin(\theta)^2\tilde{\mu}^2_{2,1}\right]\mathrm{ad}_f\tilde{g}_2.
\end{aligned}
$$

And finally,

$$
\begin{aligned}
[g_2, \mathrm{ad}_f g_1] &= \left[\lambda\left(-\tilde{g}_1 \sin(\theta) + \tilde{g}_2 \cos(\theta)\right), \lambda\left(\cos(\theta)\mathrm{ad}_f\tilde{g}_1 + \sin(\theta)\mathrm{ad}_f\tilde{g}_2\right)\right], \\
&= \lambda^2 \left[-\tilde{g}_1 \sin(\theta) + \tilde{g}_2 \cos(\theta), \cos(\theta)\mathrm{ad}_f\tilde{g}_1 + \sin(\theta)\mathrm{ad}_f\tilde{g}_2\right] + \frac{1}{\lambda}\mathrm{L}_{g_2}(\lambda)\,\mathrm{ad}_f g_1, \\
&= \lambda \mathrm{L}_{g_2}(\cos(\theta))\,\mathrm{ad}_f\tilde{g}_1 + \lambda \mathrm{L}_{g_2}(\sin(\theta))\,\mathrm{ad}_f\tilde{g}_2 + \frac{1}{\lambda}\mathrm{L}_{g_2}(\lambda)\,\mathrm{ad}_f g_1 \\
&+ \lambda^2 \big(-\cos(\theta)\sin(\theta)\left[\tilde{g}_1, \mathrm{ad}_f\tilde{g}_1\right] - \sin(\theta)^2\left[\tilde{g}_1, \mathrm{ad}_f\tilde{g}_2\right]\big) \\
&+ \lambda^2 \big(\cos(\theta)^2\left[\tilde{g}_2, \mathrm{ad}_f\tilde{g}_1\right] + \cos(\theta)\sin(\theta)\left[\tilde{g}_2, \mathrm{ad}_f\tilde{g}_2\right]\big), \\
&= \lambda \mathrm{L}_{g_2}(\theta)\left(-\sin(\theta)\mathrm{ad}_f\tilde{g}_1 + \cos(\theta)\mathrm{ad}_f\tilde{g}_2\right) + \frac{1}{\lambda}\mathrm{L}_{g_2}(\lambda)\,\mathrm{ad}_f g_1 \\
&- \lambda^2 \cos(\theta)\sin(\theta)\left(\left[\tilde{g}_1, \mathrm{ad}_f\tilde{g}_1\right] - \left[\tilde{g}_2, \mathrm{ad}_f\tilde{g}_2\right]\right) \\
&- \lambda^2 \sin(\theta)^2\left[\tilde{g}_1, \mathrm{ad}_f\tilde{g}_2\right] + \lambda^2\cos(\theta)^2\left[\tilde{g}_2, \mathrm{ad}_f\tilde{g}_1\right], \\
&= \frac{1}{\lambda}\mathrm{L}_{g_2}(\lambda)\,\mathrm{ad}_f g_1 + \mathrm{L}_{g_2}(\theta)\,\mathrm{ad}_f g_2 \\
&+ \lambda^2\left[-\frac{1}{2}\sin(2\theta)\tilde{\mu}^1 - \sin(\theta)^2\tilde{\mu}^1_{1,2} + \cos(\theta)^2\tilde{\mu}^1_{2,1}\right]\mathrm{ad}_f\tilde{g}_1 \\
&+ \lambda^2\left[-\frac{1}{2}\sin(2\theta)\tilde{\mu}^2 - \sin(\theta)^2\tilde{\mu}^2_{1,2} + \cos(\theta)^2\tilde{\mu}^2_{2,1}\right]\mathrm{ad}_f\tilde{g}_2.
\end{aligned}
$$

Collecting the results of those three computations and using

$$
\begin{pmatrix} \mathrm{ad}_f\tilde{g}_1 \\ \mathrm{ad}_f\tilde{g}_2 \end{pmatrix} = \frac{1}{\lambda}\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} \mathrm{ad}_f g_1 \\ \mathrm{ad}_f g_2 \end{pmatrix}
$$

we obtain the required equality.

4.B Detailed computations for the proof of Theorem 4.4

We detail the computations in the proof (pE3) $\Longrightarrow$ (pE4) of Theorem 4.4. Assume that (g_1, g_2) is a weak orthonormal frame with structure functions $\mu^k_{i,j}$ and μ^k satisfying (4.6) and (4.8). We show that $(\tilde g_1, \tilde g_2) = (g_1, g_2)\beta\left(\frac{1}{\lambda}, -\theta\right)$, where λ and θ are smooth solutions of the systems (4.7), is a strong orthonormal frame. To see that, we apply formula (4.3) on the transformation $(\tilde g_1, \tilde g_2)\beta(\lambda, \theta) = (g_1, g_2)$:

$$\begin{pmatrix} \mu^1 & \mu^2 \\ \mu^1_{1,2} & \mu^2_{1,2} \\ \mu^1_{2,1} & \mu^2_{2,1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda}\mathrm{L}_{g_1}(\lambda) + \mathrm{L}_{g_2}(\theta) & \mathrm{L}_{g_1}(\theta) - \frac{1}{\lambda}\mathrm{L}_{g_2}(\lambda) \\ -\mathrm{L}_{g_1}(\theta) & \frac{1}{\lambda}\mathrm{L}_{g_1}(\lambda) \\ \frac{1}{\lambda}\mathrm{L}_{g_2}(\lambda) & \mathrm{L}_{g_2}(\theta) \end{pmatrix}$$
$$+ \begin{pmatrix} \cos(2\theta) & \sin(2\theta) & \sin(2\theta) \\ -\frac{1}{2}\sin(2\theta) & \cos(\theta)^2 & -\sin(\theta)^2 \\ -\frac{1}{2}\sin(2\theta) & -\sin(\theta)^2 & \cos(\theta)^2 \end{pmatrix} \begin{pmatrix} \tilde\mu^1 & \tilde\mu^2 \\ \tilde\mu^1_{1,2} & \tilde\mu^2_{1,2} \\ \tilde\mu^1_{2,1} & \tilde\mu^2_{2,1} \end{pmatrix} \beta(\lambda, \theta).$$

Using $\frac{1}{\lambda}\mathrm{L}_{g_1}(\lambda) = \mu^2_{1,2}$, $\frac{1}{\lambda}\mathrm{L}_{g_2}(\lambda) = \mu^1_{2,1}$, $\mathrm{L}_{g_1}(\theta) = -\mu^1_{1,2}$, and $\mathrm{L}_{g_2}(\theta) = \mu^2_{2,1}$, we obtain

$$\begin{pmatrix} \mu^1 & \mu^2 \\ \mu^1_{1,2} & \mu^2_{1,2} \\ \mu^1_{2,1} & \mu^2_{2,1} \end{pmatrix} = \begin{pmatrix} \overbrace{\mu^2_{1,2} + \mu^2_{2,1}}^{\mu^1} & \overbrace{-\mu^1_{1,2} - \mu^1_{2,1}}^{\mu^2} \\ \mu^1_{1,2} & \mu^2_{1,2} \\ \mu^1_{2,1} & \mu^2_{2,1} \end{pmatrix}$$
$$+ \begin{pmatrix} \cos(2\theta) & \sin(2\theta) & \sin(2\theta) \\ -\frac{1}{2}\sin(2\theta) & \cos(\theta)^2 & -\sin(\theta)^2 \\ -\frac{1}{2}\sin(2\theta) & -\sin(\theta)^2 & \cos(\theta)^2 \end{pmatrix} \begin{pmatrix} \tilde\mu^1 & \tilde\mu^2 \\ \tilde\mu^1_{1,2} & \tilde\mu^2_{1,2} \\ \tilde\mu^1_{2,1} & \tilde\mu^2_{2,1} \end{pmatrix} \beta\left(\frac{1}{\lambda}, -\theta\right).$$

Thus, we conclude

$$\begin{pmatrix} \cos(2\theta) & \sin(2\theta) & \sin(2\theta) \\ -\frac{1}{2}\sin(2\theta) & \cos(\theta)^2 & -\sin(\theta)^2 \\ -\frac{1}{2}\sin(2\theta) & -\sin(\theta)^2 & \cos(\theta)^2 \end{pmatrix} \begin{pmatrix} \tilde\mu^1 & \tilde\mu^2 \\ \tilde\mu^1_{1,2} & \tilde\mu^2_{1,2} \\ \tilde\mu^1_{2,1} & \tilde\mu^2_{2,1} \end{pmatrix} \beta\left(\frac{1}{\lambda}, -\theta\right) = 0,$$

implying $\tilde\mu^k_{i,j} = \tilde\mu^k = 0$.

4.C Detailed computations for Lemma 4.1

In that appendix we show how the structure function μ^k_i, ν^k, and γ_k of a p-elliptic systems $\Xi_{pE} = (A, B_1, B_2, C)$ are related with the structure functions $\tilde\mu^k_i$, $\tilde\nu^k$, $\tilde\gamma_k$ of an equivalent p-elliptic system $\tilde\Xi_{pE} = (\tilde A, \tilde B_1, \tilde B_2, \tilde C)$. Recall that under reparametrisations $w = \alpha + \beta(\lambda, \theta)\tilde w$ we have the following relation between the p-elliptic systems Ξ_{pE} and $\tilde\Xi_{pE}$:

$$\begin{pmatrix} \tilde A \\ \tilde B_1 \\ \tilde B_2 \end{pmatrix} = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 2\lambda(\alpha_1\cos(\theta) + \alpha_2\sin(\theta)) & \lambda\cos(\theta) & \lambda\sin(\theta) \\ 2\lambda(-\alpha_1\sin(\theta) + \alpha_2\cos(\theta)) & -\lambda\sin(\theta) & \lambda\cos(\theta) \end{pmatrix} \begin{pmatrix} A \\ B_1 \\ B_2 \end{pmatrix},$$

equivalently
$$\begin{pmatrix} A \\ B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda^2} & 0 & 0 \\ -2\frac{\alpha_1}{\lambda^2} & \frac{\cos\theta}{\lambda} & -\frac{\sin\theta}{\lambda} \\ -2\frac{\alpha_2}{\lambda^2} & \frac{\sin\theta}{\lambda} & \frac{\cos\theta}{\lambda} \end{pmatrix} \begin{pmatrix} \tilde A \\ \tilde B_1 \\ \tilde B_2 \end{pmatrix}.$$

Computations have been made by hand and verified with the symbolic computer environment Maple. We begin by showing relation (4.11) for the structure functions μ_i^k.

$$
\begin{aligned}
\tilde{\mu}_1^0 \tilde{A} + \tilde{\mu}_1^1 \tilde{B}_1 + \tilde{\mu}_1^2 \tilde{B}_2 = \left[\tilde{A}, \tilde{B}_1\right] &= \left[\lambda^2 A, 2\lambda(\alpha_1 \cos(\theta) + \alpha_2 \sin(\theta))A + \lambda\cos(\theta) B_1 + \lambda \sin(\theta) B_2\right], \\
&= -\mathrm{L}_{\tilde{B}_1}\left(\lambda^2\right) A + \lambda^3 \cos(\theta)\left[A, B_1\right] + \lambda^3 \sin(\theta)\left[A, B_2\right] + \lambda^2 \mathrm{L}_A\left(\lambda\cos(\theta)\right) B_1 \\
&\quad + \lambda^2 \mathrm{L}_A\left(\lambda \sin(\theta)\right) B_2 + 2\lambda^2 \mathrm{L}_A\left(\lambda(\alpha_1\cos(\theta) + \alpha_2\sin(\theta))\right) A, \\
&= \left[-\mathrm{L}_{\tilde{B}_1}\left(\lambda^2\right) + 2\lambda^2 \mathrm{L}_A\left(\lambda(\alpha_1\cos(\theta)+\alpha_2\sin(\theta))\right) + \lambda^3\cos(\theta)\mu_1^0 + \lambda^3 \sin(\theta)\mu_2^0\right] A \\
&\quad + \left[\lambda^2 \mathrm{L}_A\left(\lambda\cos(\theta)\right) + \lambda^3\cos(\theta)\mu_1^1 + \lambda^3\sin(\theta)\mu_2^1\right] B_1 \\
&\quad + \left[\lambda^2 \mathrm{L}_A\left(\lambda\sin(\theta)\right) + \lambda^3\cos(\theta)\mu_1^2 + \lambda^3\sin(\theta)\mu_2^2\right] B_2,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\tilde{\mu}_1^0 = &\frac{1}{\lambda^2}\left[-\mathrm{L}_{\tilde{B}_1}\left(\lambda^2\right) + 2\lambda^2 \mathrm{L}_A\left(\lambda(\alpha_1\cos(\theta)+\alpha_2\sin(\theta))\right) + \lambda^3\cos(\theta)\mu_1^0 + \lambda^3 \sin(\theta)\mu_2^0\right] \\
&- 2\frac{\alpha_1}{\lambda^2}\left[\lambda^2 \mathrm{L}_A\left(\lambda\cos(\theta)\right) + \lambda^3\cos(\theta)\mu_1^1 + \lambda^3\sin(\theta)\mu_2^1\right] \\
&- 2\frac{\alpha_2}{\lambda^2}\left[\lambda^2 \mathrm{L}_A\left(\lambda\sin(\theta)\right) + \lambda^3\cos(\theta)\mu_1^2 + \lambda^3\sin(\theta)\mu_2^2\right], \\
= &-2\mathrm{L}_{\tilde{B}_1}\left(\Lambda\right) + 2\mathrm{L}_A\left(\lambda\right)\left(\alpha_1\cos(\theta)+\alpha_2\sin(\theta)\right) + 2\lambda\mathrm{L}_A\left(\alpha_1\cos(\theta)+\alpha_2\sin(\theta)\right) \\
&+ \lambda\cos(\theta)\mu_1^0 + \lambda\sin(\theta)\mu_2^0 \\
&- 2\alpha_1\lambda\left[\mathrm{L}_A\left(\Lambda\right)\cos(\theta) - \sin(\theta)\mathrm{L}_A\left(\theta\right) + \cos(\theta)\mu_1^1 + \sin(\theta)\mu_2^1\right] \\
&- 2\alpha_2\lambda\left[\mathrm{L}_A\left(\Lambda\right)\sin(\theta) + \cos(\theta)\mathrm{L}_A\left(\theta\right) + \cos(\theta)\mu_1^2 + \sin(\theta)\mu_2^2\right], \\
= &-2\mathrm{L}_{\tilde{B}_1}\left(\Lambda\right) + \lambda\cos(\theta)\left(\mu_1^0 - 2\alpha_1\mu_1^1 - 2\alpha_2\mu_1^2 + 2\mathrm{L}_A\left(\alpha_1\right)\right) \\
&+ \lambda\sin(\theta)\left(\mu_2^0 + 2\alpha_1\mu_2^1 - 2\alpha_2\mu_2^2 + 2\mathrm{L}_A\left(\alpha_2\right)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\mu}_1^1 &= \frac{\cos(\theta)}{\lambda}\left[\lambda^2 \mathrm{L}_A\left(\lambda\cos(\theta)\right) + \lambda^3\cos(\theta)\mu_1^1 + \lambda^3\sin(\theta)\mu_2^1\right] \\
&\quad + \frac{\sin(\theta)}{\lambda}\left[\lambda^2 \mathrm{L}_A\left(\lambda\sin(\theta)\right) + \lambda^3\cos(\theta)\mu_1^2 + \lambda^3\sin(\theta)\mu_2^2\right], \\
&= \lambda^2\cos(\theta)\left[\mathrm{L}_A\left(\Lambda\right)\cos(\theta) - \sin(\theta)\mathrm{L}_A\left(\theta\right) + \cos(\theta)\mu_1^1 + \sin(\theta)\mu_2^1\right] \\
&\quad + \lambda^2\sin(\theta)\left[\mathrm{L}_A\left(\Lambda\right)\sin(\theta) + \cos(\theta)\mathrm{L}_A\left(\theta\right) + \cos(\theta)\mu_1^2 + \sin(\theta)\mu_2^2\right], \\
&= \lambda^2\left[\mathrm{L}_A\left(\Lambda\right) + \cos(\theta^2\mu_1^1 + \sin(\theta)^2\mu_2^2 + \cos(\theta)\sin(\theta)(\mu_2^1 + \mu_1^2)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\mu}_1^2 &= -\frac{\sin(\theta)}{\lambda}\left[\lambda^2 \mathrm{L}_A\left(\lambda\cos(\theta)\right) + \lambda^3\cos(\theta)\mu_1^1 + \lambda^3\sin(\theta)\mu_2^1\right] \\
&\quad + \frac{\cos(\theta)}{\lambda}\left[\lambda^2 \mathrm{L}_A\left(\lambda\sin(\theta)\right) + \lambda^3\cos(\theta)\mu_1^2 + \lambda^3\sin(\theta)\mu_2^2\right], \\
&= -\lambda^2\sin(\theta)\left[\mathrm{L}_A\left(\Lambda\right)\cos(\theta) - \sin(\theta)\mathrm{L}_A\left(\theta\right) + \cos(\theta)\mu_1^1 + \sin(\theta)\mu_2^1\right] \\
&\quad + \lambda^2\cos(\theta)\left[\mathrm{L}_A\left(\Lambda\right)\sin(\theta) + \cos(\theta)\mathrm{L}_A\left(\theta\right) + \cos(\theta)\mu_1^2 + \sin(\theta)\mu_2^2\right], \\
&= \lambda^2\left[\mathrm{L}_A\left(\theta\right) + \cos(\theta)^2\mu_1^2 - \sin(\theta)^2\mu_2^1 - \cos(\theta)\sin(\theta)(\mu_1^1 - \mu_2^2)\right].
\end{aligned}
$$

The same type of calculations is done for $\left[\tilde{A}, \tilde{B}_2\right]$. Next we show relation (4.12) for the structure functions ν^k.

$$
\begin{aligned}
\tilde{\nu}_1^0 \tilde{A} + \tilde{\nu}_1^1 \tilde{B}_1 + \tilde{\nu}_1^2 \tilde{B}_2 = \left[\tilde{B}_1, \tilde{B}_2\right], & \\
= [2\lambda(\alpha_1 \cos(\theta) + \alpha_2 \sin(\theta))A + \lambda \cos(\theta) B_1 + \lambda \sin(\theta) B_2 & \\
2\lambda(-\alpha_1 \sin(\theta) + \alpha_2 \cos(\theta))A - \lambda \sin(\theta) B_1 + \lambda \cos(\theta) B_2], &
\end{aligned}
$$

which yields

$$
\begin{aligned}
\left[\tilde{B}_1, \tilde{B}_2\right] = & -2\lambda^2 \alpha_2 [A, B_1] + 2\lambda^2 \alpha_1 [A, B_2] + \lambda^2 [B_1, B_2] \\
& + 2\lambda \big[2(\alpha_1 \cos(\theta) + \alpha_2 \sin(\theta)) \mathrm{L}_A (\lambda(-\alpha_1 \sin(\theta) + \alpha_2 \cos(\theta))) \\
& \quad - 2(-\alpha_1 \sin(\theta) + \alpha_2 \cos(\theta)) \mathrm{L}_A (\lambda(\alpha_1 \cos(\theta) + \alpha_2 \sin(\theta))) \\
& \quad + \sin(\theta) \mathrm{L}_{B_1} (\lambda(\alpha_1 \cos(\theta) + \alpha_2 \sin(\theta))) \\
& \quad - \cos(\theta) \mathrm{L}_{B_2} (\lambda(\alpha_1 \cos(\theta) + \alpha_2 \sin(\theta))) \\
& \quad + \cos(\theta) \mathrm{L}_{B_1} (\lambda(-\alpha_1 \sin(\theta) + \alpha_2 \cos(\theta))) \\
& \quad + \sin(\theta) \mathrm{L}_{B_2} (\lambda(-\alpha_1 \sin(\theta) + \alpha_2 \cos(\theta))) \big] A \\
& + \lambda \big[- 2(\alpha_1 \cos(\theta) + \alpha_2 \sin(\theta)) \mathrm{L}_A (\lambda \sin(\theta)) \\
& \quad - 2(-\alpha_1 \sin(\theta) + \alpha_2 \cos(\theta)) \mathrm{L}_A (\lambda \cos(\theta)) - \cos(\theta) \mathrm{L}_{B_1} (\lambda \sin(\theta)) \\
& \quad + \sin(\theta) \mathrm{L}_{B_1} (\lambda \cos(\theta)) - \cos(\theta) \mathrm{L}_{B_2} (\lambda \cos(\theta)) - \sin(\theta) \mathrm{L}_{B_2} (\lambda \sin(\theta)) \big] B_1 \\
& + \lambda \big[2(\alpha_1 \cos(\theta) + \alpha_2 \sin(\theta)) \mathrm{L}_A (\lambda \cos(\theta)) \\
& \quad - 2(-\alpha_1 \sin(\theta) + \alpha_2 \cos(\theta)) \mathrm{L}_A (\lambda \sin(\theta)) + \cos(\theta) \mathrm{L}_{B_1} (\lambda \cos(\theta)) \\
& \quad + \sin(\theta) \mathrm{L}_{B_1} (\lambda \sin(\theta)) + \sin(\theta) \mathrm{L}_{B_2} (\lambda \cos(\theta)) - \cos(\theta) \mathrm{L}_{B_2} (\lambda \sin(\theta)) \big] B_2.
\end{aligned}
$$

Hence, we deduce first

$$
\begin{aligned}
\tilde{\nu}^0 = & -2\alpha_2 \mu_1^0 + 2\alpha_1 \mu_2^0 + \nu^0 \\
& + \frac{2}{\lambda} \big[2(\alpha_1 \cos(\theta) + \alpha_2 \sin(\theta)) \mathrm{L}_A (\lambda(-\alpha_1 \sin(\theta) + \alpha_2 \cos(\theta))) \\
& - 2(-\alpha_1 \sin(\theta) + \alpha_2 \cos(\theta)) \mathrm{L}_A (\lambda(\alpha_1 \cos(\theta) + \alpha_2 \sin(\theta))) \\
& + \sin(\theta) \mathrm{L}_{B_1} (\lambda(\alpha_1 \cos(\theta) + \alpha_2 \sin(\theta))) - \cos(\theta) \mathrm{L}_{B_2} (\lambda(\alpha_1 \cos(\theta) + \alpha_2 \sin(\theta))) \\
& + \cos(\theta) \mathrm{L}_{B_1} (\lambda(-\alpha_1 \sin(\theta) + \alpha_2 \cos(\theta))) + \sin(\theta) \mathrm{L}_{B_2} (\lambda(-\alpha_1 \sin(\theta) + \alpha_2 \cos(\theta))) \big] \\
& - 2\frac{\alpha_1}{\lambda} \big[- 2\lambda^2 \alpha_2 \mu_1^1 + 2\lambda^2 \alpha_1 \mu_2^1 + \lambda^2 \nu^1 - 2(\alpha_1 \cos(\theta) + \alpha_2 \sin(\theta)) \mathrm{L}_A (\lambda \sin(\theta)) \\
& - 2(-\alpha_1 \sin(\theta) + \alpha_2 \cos(\theta)) \mathrm{L}_A (\lambda \cos(\theta)) - \cos(\theta) \mathrm{L}_{B_1} (\lambda \sin(\theta)) \\
& + \sin(\theta) \mathrm{L}_{B_1} (\lambda \cos(\theta)) - \cos(\theta) \mathrm{L}_{B_2} (\lambda \cos(\theta)) - \sin(\theta) \mathrm{L}_{B_2} (\lambda \sin(\theta)) \big] \\
& - 2\frac{\alpha_2}{\lambda} \big[- 2\lambda^2 \alpha_2 \mu_1^2 + 2\lambda^2 \alpha_1 \mu_2^2 + \lambda^2 \nu^2 + 2(\alpha_1 \cos(\theta) + \alpha_2 \sin(\theta)) \mathrm{L}_A (\lambda \cos(\theta)) \\
& - 2(-\alpha_1 \sin(\theta) + \alpha_2 \cos(\theta)) \mathrm{L}_A (\lambda \sin(\theta)) + \cos(\theta) \mathrm{L}_{B_1} (\lambda \cos(\theta)) \\
& + \sin(\theta) \mathrm{L}_{B_1} (\lambda \sin(\theta)) + \sin(\theta) \mathrm{L}_{B_2} (\lambda \cos(\theta)) - \cos(\theta) \mathrm{L}_{B_2} (\lambda \sin(\theta)) \big],
\end{aligned}
$$

implying

$$
\begin{aligned}
\tilde{\nu}^0 = \nu^0 &- 2\alpha_1\nu^1 - 2\alpha_2\nu^2 - 2\alpha_2\mu_1^0 + 2\alpha_1\mu_2^0 \\
&+ 4\alpha_1 \mathrm{L}_A(\alpha_2) - 4\alpha_2\mathrm{L}_A(\alpha_1) - 4((\alpha_1)^2 + (\alpha_2)^2)\mathrm{L}_A(\theta) \\
&+ 2\alpha_2\mathrm{L}_{B_1}(\Lambda) - 2\alpha_1\mathrm{L}_{B_2}(\Lambda) + 2\mathrm{L}_{B_1}(\alpha_2) - 2\mathrm{L}_{B_2}(\alpha_1) - 2\alpha_1\mathrm{L}_{B_1}(\theta) - 2\alpha_2\mathrm{L}_{B_2}(\theta) \\
&+ 4\alpha_1\alpha_2\mathrm{L}_A(\Lambda) + 4(\alpha_1)^2\mathrm{L}_A(\theta) + 2\alpha_1\mathrm{L}_{B_1}(\theta) + 2\alpha_1\mathrm{L}_{B_2}(\Lambda) \\
&- 4\alpha_1\alpha_2\mathrm{L}_A(\Lambda) + 4(\alpha_2)^2\mathrm{L}_A(\theta) + 2\alpha_2\mathrm{L}_{B_2}(\theta) - 2\alpha_2\mathrm{L}_{B_1}(\Lambda), \\
= \nu^0 &- 2\alpha_1\nu^1 - 2\alpha_2\nu^2 - 2\alpha_2\mu_1^0 + 2\alpha_1\mu_2^0 + 4\alpha_1\alpha_2(\mu_1^1 - \mu_2^2) + 4(\alpha_2)^2\mu_1^2 - 4(\alpha_1)^2\mu_2^1 \\
&+ 4\alpha_1\mathrm{L}_A(\alpha_2) - 4\alpha_2\mathrm{L}_A(\alpha_1) + 2\mathrm{L}_{B_1}(\alpha_2) - 2\mathrm{L}_{B_2}(\alpha_1).
\end{aligned}
$$

Second

$$
\begin{aligned}
\tilde{\nu}^1 = \cos(\theta)\big[&- 2\lambda\alpha_2\mu_1^1 + 2\lambda\alpha_1\mu_2^1 + \lambda\nu^1 - 2(\alpha_1\cos(\theta) + \alpha_2\sin(\theta))\mathrm{L}_A(\lambda\sin(\theta)) \\
&- 2(-\alpha_1\sin(\theta) + \alpha_2\cos(\theta))\mathrm{L}_A(\lambda\cos(\theta)) - \cos(\theta)\mathrm{L}_{B_1}(\lambda\sin(\theta)) \\
&+ \sin(\theta)\mathrm{L}_{B_1}(\lambda\cos(\theta)) - \cos(\theta)\mathrm{L}_{B_1}(\lambda\cos(\theta)) - \sin(\theta)\mathrm{L}_{B_2}(\lambda\sin(\theta))\big] \\
+ \sin(\theta)\big[&- 2\lambda\alpha_2\mu_1^2 + 2\lambda\alpha_1\mu_2^2 + \lambda\nu^2 + 2(\alpha_1\cos(\theta) + \alpha_2\sin(\theta))\mathrm{L}_A(\lambda\cos(\theta)) \\
&- 2(-\alpha_1\sin(\theta) + \alpha_2\cos(\theta))\mathrm{L}_A(\lambda\sin(\theta)) + \cos(\theta)\mathrm{L}_{B_1}(\lambda\cos(\theta)) \\
&+ \sin(\theta)\mathrm{L}_{B_1}(\lambda\sin(\theta)) + \sin(\theta)\mathrm{L}_{b_2}(\lambda\cos(\theta)) - \cos(\theta)\mathrm{L}_{B_2}(\lambda\sin(\theta))\big], \\
= \lambda\cos(\theta)\big[&\nu^1 - 2\alpha_2\mu_1^1 + 2\alpha_1\mu_2^1 - 2\alpha_2\mathrm{L}_A(\Lambda) - 2\alpha_1\mathrm{L}_A(\theta) - \mathrm{L}_{B_1}(\theta) - \mathrm{L}_{B_2}(\theta)\big] \\
+ \lambda\sin(\theta)\big[&\nu^2 - 2\alpha_2\mu_1^2 + 2\alpha_1\mu_2^2 + 2\alpha_1\mathrm{L}_A(\Lambda) - 2\alpha_2\mathrm{L}_A(\theta) + \mathrm{L}_{B_1}(\Lambda) - \mathrm{L}_{B_2}(\theta)\big].
\end{aligned}
$$

And, third

$$
\begin{aligned}
\tilde{\nu}^2 = -\sin(\theta)\big[&- 2\lambda\alpha_2\mu_1^1 + 2\lambda\alpha_1\mu_2^1 + \lambda\nu^1 - 2(\alpha_1\cos(\theta) + \alpha_2\sin(\theta))\mathrm{L}_A(\lambda\sin(\theta)) \\
&- 2(-\alpha_1\sin(\theta) + \alpha_2\cos(\theta))\mathrm{L}_A(\lambda\cos(\theta)) - \cos(\theta)\mathrm{L}_{B_1}(\lambda\sin(\theta)) \\
&+ \sin(\theta)\mathrm{L}_{B_1}(\lambda\cos(\theta)) - \cos(\theta)\mathrm{L}_{B_1}(\lambda\cos(\theta)) - \sin(\theta)\mathrm{L}_{B_2}(\lambda\sin(\theta))\big] \\
+ \cos(\theta)\big[&- 2\lambda\alpha_2\mu_1^2 + 2\lambda\alpha_1\mu_2^2 + \lambda\nu^2 + 2(\alpha_1\cos(\theta) + \alpha_2\sin(\theta))\mathrm{L}_A(\lambda\cos(\theta)) \\
&- 2(-\alpha_1\sin(\theta) + \alpha_2\cos(\theta))\mathrm{L}_A(\lambda\sin(\theta)) + \cos(\theta)\mathrm{L}_{B_1}(\lambda\cos(\theta)) \\
&+ \sin(\theta)\mathrm{L}_{B_1}(\lambda\sin(\theta)) + \sin(\theta)\mathrm{L}_{b_2}(\lambda\cos(\theta)) - \cos(\theta)\mathrm{L}_{B_2}(\lambda\sin(\theta))\big], \\
= -\lambda\sin(\theta)\big[&\nu^1 - 2\alpha_2\mu_1^1 + 2\alpha_1\mu_2^1 - 2\alpha_2\mathrm{L}_A(\Lambda) - 2\alpha_1\mathrm{L}_A(\theta) - \mathrm{L}_{B_1}(\theta) - \mathrm{L}_{B_2}(\theta)\big] \\
+ \lambda\cos(\theta)\big[&\nu^2 - 2\alpha_2\mu_1^2 + 2\alpha_1\mu_2^2 + 2\alpha_1\mathrm{L}_A(\Lambda) - 2\alpha_2\mathrm{L}_A(\theta) + \mathrm{L}_{B_1}(\Lambda) - \mathrm{L}_{B_2}(\theta)\big].
\end{aligned}
$$

We now compute the transformation of the structure functions γ^k:

$$
\begin{aligned}
\tilde{\gamma}^0\tilde{A} + \tilde{\gamma}^1\tilde{B}_1 + \tilde{\gamma}^2\tilde{B}_2 = \tilde{C} &= C + \left((\alpha_1)^2 + (\alpha_2)^2\right)A + \alpha_1 B_1 + \alpha_2 B_2, \\
&= \left(\gamma^0 + (\alpha_1)^2 + (\alpha_2)^2\right)A + \left(\gamma^1 + \alpha_1\right)B_1 + \left(\gamma^2 + \alpha^2\right)B_2, \\
&= \frac{1}{\lambda^2}\left(\gamma^0 + (\alpha_1)^2 + (\alpha_2)^2 - 2\alpha_1(\gamma^1 + \alpha_1) - 2\alpha_2(\gamma^2 + \alpha^2)\right)\tilde{A} \\
&\quad + \frac{1}{\lambda}\left(\cos(\theta)(\gamma^1 + \alpha_1) - \sin(\theta)(\gamma^2 + \alpha^2)\right)\tilde{B}_1 \\
&\quad + \frac{1}{\lambda}\left(\sin(\theta)(\gamma^1 + \alpha_1) + \cos(\theta)(\gamma^2 + \alpha^2)\right)\tilde{B}_2 \\
&= \frac{1}{\lambda^2}\left(\gamma^0 - (\alpha_1)^2 - (\alpha_2)^2 - 2\alpha_1\gamma^1 - 2\alpha_2\gamma^2\right)\tilde{A} \\
&\quad + \frac{1}{\lambda}\left(\cos(\theta)(\gamma^1 + \alpha_1) - \sin(\theta)(\gamma^2 + \alpha^2)\right)\tilde{B}_1 \\
&\quad + \frac{1}{\lambda}\left(\sin(\theta)(\gamma^1 + \alpha_1) + \cos(\theta)(\gamma^2 + \alpha^2)\right)\tilde{B}_2.
\end{aligned}
$$

And finally relation (4.14) follows from the following application of the Jacobi identity

$$
\begin{aligned}
[A,[B_1,B_2]] &= [B_1,[A,B_2]] - [B_2,[A,B_1]]\,,\\
\left[A,\nu^0 A+\nu^1 B_1+\nu^2 B_2\right] &= \left[B_1,\mu_2^0 A+\mu_2^1 B_1+\mu_2^2 B_2\right] - \left[B_2,\mu_1^0 A+\mu_1^1 B_1+\mu_1^2 B_2\right]\,,\\
\left(\mathrm{L}_A\left(\nu^0\right)+\nu^1\mu_1^0\nu^2\mu_2^0\right)A + \left(\mathrm{L}_A\left(\nu^1\right)+\nu^1\mu_1^1\nu^2\mu_2^1\right)B_1 &+ \left(\mathrm{L}_A\left(\nu^2\right)+\nu^1\mu_1^2\nu^2\mu_2^2\right)B_2 =\\
&\left(\nu^0(\mu_1^1+\mu_2^2)+\mathrm{L}_{B_1}\left(\mu_2^0\right)-\mathrm{L}_{B_2}\left(\mu_1^0\right)\right)A\\
&+\left(\nu^1(\mu_1^1+\mu_2^2)+\mu_1^0\mu_2^1-\mu_2^0\mu_1^1+\mathrm{L}_{B_1}\left(\mu_2^1\right)-\mathrm{L}_{B_2}\left(\mu_1^1\right)\right)B_1\\
&+\left(\nu^2(\mu_1^1+\mu_2^2)+\mu_1^0\mu_2^2-\mu_2^0\mu_1^2+\mathrm{L}_{B_1}\left(\mu_2^2\right)-\mathrm{L}_{B_2}\left(\mu_1^2\right)\right)B_2.
\end{aligned}
$$

4.D Transformation of the structure functions attached to a WIF

Let $(\tilde g_1,\tilde g_2)$ be a weak p-hyperbolic frame with structure functions $\tilde\mu_{i,i}^j$ and $\tilde\mu^j$. Then the structure functions $\mu_{i,i}^j$ and μ^j of $(g_1,g_2)=(\beta_1\tilde g_1,\beta_2\tilde g_2)$ are given by the following formulae (each equation is considered modulo $\mathcal{D}^0$). First,

$$
\begin{aligned}
\mu_{1,1}^1\mathrm{ad}_f g_1+\mu_{1,1}^2\mathrm{ad}_f g_2 &= [g_1,\mathrm{ad}_f g_1] = (\beta_1)^2\,[\tilde g_1,\mathrm{ad}_f\tilde g_1] + \beta_1\mathrm{L}_{\tilde g_1}(\beta_1)\,\mathrm{ad}_f\tilde g_1,\\
&= \left((\beta_1)^2\tilde\mu_{1,1}^1+\beta_1\mathrm{L}_{\tilde g_1}(\beta_1)\right)\mathrm{ad}_f\tilde g_1 + (\beta_1)^2\tilde\mu_{1,1}^2\mathrm{ad}_f\tilde g_2,\\
&= \left((\beta_1)\tilde\mu_{1,1}^1+\mathrm{L}_{\tilde g_1}(\beta_1)\right)\mathrm{ad}_f g_1 + \frac{(\beta_1)^2}{\beta_2}\tilde\mu_{1,1}^2\mathrm{ad}_f g_2.
\end{aligned}
$$

Second,

$$
\begin{aligned}
\mu_{2,2}^1\mathrm{ad}_f g_1+\mu_{2,2}^2\mathrm{ad}_f g_2 &= [g_2,\mathrm{ad}_f g_2] = (\beta_2)^2\,[\tilde g_2,\mathrm{ad}_f\tilde g_2] + \beta_2\mathrm{L}_{\tilde g_2}(\beta_2)\,\mathrm{ad}_f\tilde g_2,\\
&= (\beta_2)^2\tilde\mu_{2,2}^1\mathrm{ad}_f\tilde g_1 + \left((\beta_2)^2\tilde\mu_{2,2}^2+\beta_2\mathrm{L}_{\tilde g_2}(\beta_2)\right)\mathrm{ad}_f\tilde g_2,\\
&= \frac{(\beta_2)^2}{\beta_1}\tilde\mu_{2,2}^1\mathrm{ad}_f g_1 + \left(\beta_2\tilde\mu_{2,2}^2+\mathrm{L}_{\tilde g_2}(\beta_2)\right)\mathrm{ad}_f g_2.
\end{aligned}
$$

And, third

$$
\begin{aligned}
\mu^1\mathrm{ad}_f g_1+\mu^2\mathrm{ad}_f g_2 &= [g_1,\mathrm{ad}_f g_2]-[g_2,\mathrm{ad}_f g_1] = \beta_1\beta_2\left([\tilde g_1,\mathrm{ad}_f\tilde g_2]-[\tilde g_2,\mathrm{ad}_f\tilde g_1]\right)\\
&\quad+\beta_1\mathrm{L}_{\tilde g_1}(\beta_2)\,\mathrm{ad}_f\tilde g_2-\beta_2\mathrm{L}_{\tilde g_2}(\beta_1)\,\mathrm{ad}_f\tilde g_1,\\
&= \left(\beta_1\beta_2\tilde\mu^1-\beta_2\mathrm{L}_{\tilde g_2}(\beta_1)\right)\mathrm{ad}_f\tilde g_1+\left(\beta_1\beta_2\tilde\mu^2+\beta_1\mathrm{L}_{\tilde g_1}(\beta_2)\right)\mathrm{ad}_f\tilde g_2,\\
&= \left(\beta_2\tilde\mu^1-\frac{\beta_2}{\beta_1}\mathrm{L}_{\tilde g_2}(\beta_1)\right)\mathrm{ad}_f g_1+\left(\beta_1\tilde\mu^2+\frac{\beta_1}{\beta_2}\mathrm{L}_{\tilde g_1}(\beta_2)\right)\mathrm{ad}_f g_2.
\end{aligned}
$$

4.E Detailed computations for the proof of Theorem 4.10

We detail the calculations of the proof (pH3)⇒(pH4) of Theorem 4.10. Let β_1 and β_2 be solutions of

$$\left\{\begin{array}{ll} \mathrm{L}_{g_1}(\ln(\beta_1)) & = \mu^1_{1,1} \\ \mathrm{L}_{g_2}(\ln(\beta_1)) & = -\mu^1 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{ll} \mathrm{L}_{g_1}(\ln(\beta_2)) & = \mu^2 \\ \mathrm{L}_{g_2}(\ln(\beta_2)) & = \mu^2_{2,2} \end{array}\right.,$$

notice that necessarily such solutions satisfy $\beta_i(\cdot) \neq 0$. Apply the feedback $\tilde{f} = f$ and $(\tilde{g}_1, \tilde{g}_2) = \left(\frac{1}{\beta_1} g_1, \frac{1}{\beta_2} g_2\right)$, we claim that the pair $(\tilde{g}_1, \tilde{g}_2)$ is a strong p-hyperbolic frame. Indeed, modulo $\mathcal{D}^0$ we have,

$$\begin{aligned}
\left[\tilde{g}_1, \mathrm{ad}_{\tilde{f}}\tilde{g}_1\right] &= \frac{1}{(\beta_1)^2}\left[g_1, \mathrm{ad}_f g_1\right] + \frac{1}{\beta_1}\mathrm{L}_{g_1}\left(\frac{1}{\beta_1}\right)\mathrm{ad}_f g_1 \\
&= \frac{1}{(\beta_1)^2}\mu^1_{1,1}\mathrm{ad}_f g_1 - \frac{1}{(\beta_1)^3}\mathrm{L}_{g_1}(\beta_1)\,\mathrm{ad}_f g_1 = 0, \\
\left[\tilde{g}_2, \mathrm{ad}_{\tilde{f}}\tilde{g}_2\right] &= \frac{1}{(\beta_2)^2}\left[g_2, \mathrm{ad}_f g_2\right] + \frac{1}{\beta_2}\mathrm{L}_{g_2}\left(\frac{1}{\beta_2}\right)\mathrm{ad}_f g_2 \\
&= \frac{1}{(\beta_2)^2}\mu^2_{2,2}\mathrm{ad}_f g_2 - \frac{1}{(\beta_2)^3}\mathrm{L}_{g_2}(\beta_2)\,\mathrm{ad}_f g_2 = 0, \\
\left[\tilde{g}_1, \mathrm{ad}_{\tilde{f}}\tilde{g}_2\right] - \left[\tilde{g}_2, \mathrm{ad}_{\tilde{f}}\tilde{g}_1\right] &= \frac{1}{\beta_1\beta_2}\left(\left[g_1, \mathrm{ad}_f g_2\right] - \left[g_2, \mathrm{ad}_f g_1\right]\right) \\
&- \frac{1}{\beta_2}\mathrm{L}_{g_2}\left(\frac{1}{\beta_1}\right)\mathrm{ad}_f g_1 + \frac{1}{\beta_1}\mathrm{L}_{g_1}\left(\frac{1}{\beta_2}\right)\mathrm{ad}_f g_2 \\
&= \frac{1}{\beta_1\beta_2}\mu^1\mathrm{ad}_f g_1 + \frac{1}{\beta_1\beta_2}\mu^2\mathrm{ad}_f g_2 \\
&- \frac{1}{\beta_1(\beta_2)^2}\mathrm{L}_{g_2}(\beta_1)\,\mathrm{ad}_f g_1 + \frac{1}{(\beta_1)^2\beta_2}\mathrm{L}_{g_1}(\beta_2)\,\mathrm{ad}_f g_2 = 0.
\end{aligned}$$

4.F Detailed computations for Lemma 4.2

In that appendix, we show how the structure functions μ^k_i, ν, and γ_k of a weak p-hyperbolic frame are transformed under reparametrisations of the form $(w_1, w_2) = (\alpha_1 + \beta_1\tilde{w}_1, \alpha_2 + \beta_2\tilde{w}_2$. Recall from relation (4.29) that under a reparametrisation, two pH-frame (A, B_1, B_2) and $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$ are related by:

$$A = \frac{1}{\beta_1\beta_2}\tilde{A},\ B_1 = -\frac{\alpha_2}{\beta_1\beta_2}\tilde{A} + \frac{1}{\beta_1}\tilde{B}_1,\ \text{and}\ B_2 = -\frac{\alpha_1}{\beta_1\beta_2}\tilde{A} + \frac{1}{\beta_2}\tilde{B}_2.$$

We begin by showing relation (4.31) for μ^k_i.

$$\begin{aligned}
\tilde{\mu}^0_1\tilde{A} + \tilde{\mu}^1_1\tilde{B}_1 + \tilde{\mu}^2_1\tilde{B}_2 = \left[\tilde{A}, \tilde{B}_1\right] &= \left[\beta_1\beta_2 A, \alpha_2\beta_1 A + \beta_1 B_1\right], \\
&= \left(\beta_1\beta_2\mathrm{L}_A(\alpha_2\beta_1) - \alpha_2\beta_1\mathrm{L}_A(\beta_1\beta_2) - \beta_1\mathrm{L}_{B_1}(\beta_1\beta_2)\right)A \\
&+ (\beta_1)^2\beta_2\left[A, B_1\right] + \beta_1\beta_2\mathrm{L}_A(\beta_1)\,B_1, \\
&= \left((\beta_1)^2\beta_2\mu^0_1 + \beta_1\beta_2\mathrm{L}_A(\alpha_2\beta_1) - \alpha_2\beta_1\mathrm{L}_A(\beta_1\beta_2) - \beta_1\mathrm{L}_{B_1}(\beta_1\beta_2)\right)A \\
&+ \left((\beta_1)^2\beta_2\mu^1_1 + \beta_1\beta_2\mathrm{L}_A(\beta_1)\right)B_1 + (\beta_1)^2\beta_2\mu^2_1 B_2.
\end{aligned}$$

And for the bracket of $\tilde{A}$ and $\tilde{B}_2$ we have:

$$\begin{aligned}
\tilde{\mu}_2^0\tilde{A} + \tilde{\mu}_2^1\tilde{B}_1 + \tilde{\mu}_2^2\tilde{B}_2 = \left[\tilde{A}, \tilde{B}_2\right] &= \left[\beta_1\beta_2 A, \alpha_1\beta_2 A + \beta_2 B_2\right], \\
&= \left(\beta_1\beta_2 \mathrm{L}_A\left(\alpha_1\beta_2\right) - \alpha_1\beta_2\mathrm{L}_A\left(\beta_1\beta_2\right) - \beta_2\mathrm{L}_{B_2}\left(\beta_1\beta_2\right)\right) A \\
&+ \beta_1(\beta_2)^2\left[A, B_2\right] + \beta_1\beta_2\mathrm{L}_A\left(\beta_2\right) B_2, \\
&= \big(\beta_1(\beta_2)^2\mu_2^0 + \beta_1\beta_2\mathrm{L}_A\left(\alpha_1\beta_2\right) - \alpha_1\beta_2\mathrm{L}_A\left(\beta_1\beta_2\right) - \beta_2\mathrm{L}_{B_2}\left(\beta_1\beta_2\right)\big) A \\
&+ \beta_1(\beta_2)^2\mu_2^1 B_1 + \big(\beta_1(\beta_2)^2\mu_2^2 + \beta_1\beta_2\mathrm{L}_A\left(\beta_2\right)\big) B_2.
\end{aligned}$$

Thus we obtain:

$$\begin{aligned}
\tilde{\mu}_1^1 &= \beta_1\beta_2\mu_1^1 + \beta_2\mathrm{L}_A\left(\beta_1\right), \quad \tilde{\mu}_1^2 = (\beta_1)^2\mu_1^2, \\
\tilde{\mu}_1^0 &= \frac{1}{\beta_1\beta_2}\big((\beta_1)^2\beta_2\mu_1^0 + \beta_1\beta_2\mathrm{L}_A\left(\alpha_2\beta_1\right) - \alpha_2\beta_1\mathrm{L}_A\left(\beta_1\beta_2\right) - \beta_1\mathrm{L}_{B_1}\left(\beta_1\beta_2\right)\big) \\
&- \frac{\alpha_2}{\beta_1\beta_2}\big((\beta_1)^2\beta_2\mu_1^1 + \beta_1\beta_2\mathrm{L}_A\left(\beta_1\right)\big) - \frac{\alpha_1}{\beta_1\beta_2}(\beta_1)^2\beta_2\mu_1^2, \\
&= \beta_1\left(\mu_1^0 - \alpha_1\mu_1^1 - \alpha_2\mu_1^2 - \alpha_1\mathrm{L}_A\left(\ln(\beta_1)\right) - \alpha_2\mathrm{L}_A\left(\ln(\beta_2)\right) - \mathrm{L}_{B_1}\left(\ln(\beta_1\beta_2)\right) + \mathrm{L}_A\left(\alpha_2\right)\right),
\end{aligned}$$

and,

$$\begin{aligned}
\tilde{\mu}_2^1 &= (\beta_2)^2\mu_2^1, \quad \tilde{\mu}_2^2 = \beta_1\beta_2\mu_2^2 + \beta_1\mathrm{L}_A\left(\beta_2\right), \\
\tilde{\mu}_2^0 &= \frac{1}{\beta_1\beta_2}\big(\beta_1(\beta_2)^2\mu_2^0 + \beta_1\beta_2\mathrm{L}_A\left(\alpha_1\beta_2\right) - \alpha_1\beta_2\mathrm{L}_A\left(\beta_1\beta_2\right) - \beta_2\mathrm{L}_{B_2}\left(\beta_1\beta_2\right)\big) \\
&- \frac{\alpha_2}{\beta_1\beta_2}\beta_1(\beta_2)^2\mu_2^1 - \frac{\alpha_1}{\beta_1\beta_2}\big(\beta_1(\beta_2)^2\mu_2^2 + \beta_1\beta_2\mathrm{L}_A\left(\beta_2\right)\big), \\
&= \beta_2\left(\mu_2^0 - \alpha_1\mu_2^1 - \alpha_2\mu_2^2 - \alpha_2\mathrm{L}_A\left(\ln(\beta_2)\right) - \alpha_1\mathrm{L}_A\left(\ln(\beta_1)\right) - \mathrm{L}_{B_2}\left(\ln(\beta_1\beta_2)\right) + \mathrm{L}_A\left(\alpha_1\right)\right).
\end{aligned}$$

Then we show the relation (4.32) for the structure functions ν_i^k.

$$\begin{aligned}
\tilde{\nu}^0\tilde{A} + \tilde{\nu}^1\tilde{B}_1 + \tilde{\nu}^2\tilde{B}_2 = \left[\tilde{B}_1, \tilde{B}_2\right] &= \left[\alpha_2\beta_1 A + \beta_1 B_1, \alpha_1\beta_2 A + \beta_2 B_2\right], \\
&= \beta_1\beta_2\alpha_2\left[A, B_2\right] + \beta_1\beta_2\alpha_1\left[B_1, A\right] + \beta_1\beta_2\left[B_1, B_2\right] \\
&+ \left(\beta_1\alpha_2\mathrm{L}_A\left(\beta_2\alpha_1\right) - \beta_2\alpha_1\mathrm{L}_A\left(\beta_1\alpha_2\right) - \beta_2\mathrm{L}_{B_2}\left(\beta_1\alpha_2\right) + \beta_1\mathrm{L}_{B_1}\left(\beta_2\alpha_1\right)\right) A \\
&+ \left(-\beta_2\alpha_1\mathrm{L}_A\left(\beta_1\right) - \beta_2\mathrm{L}_{B_2}\left(\beta_1\right)\right) B_1 + \left(\beta_1\alpha_2\mathrm{L}_A\left(\beta_2\right) + \beta_1\mathrm{L}_{B_1}\left(\beta_2\right)\right) B_2, \\
&= \big(\beta_1\beta_2\alpha_2\mu_2^0 - \beta_1\beta_2\alpha_1\mu_1^0 + \beta_1\beta_2\nu^0 + \beta_1\alpha_2\mathrm{L}_A\left(\beta_2\alpha_1\right) \\
&-\beta_2\alpha_1\mathrm{L}_A\left(\beta_1\alpha_2\right) - \beta_2\mathrm{L}_{B_2}\left(\beta_1\alpha_2\right) + \beta_1\mathrm{L}_{B_1}\left(\beta_2\alpha_1\right)\big) A \\
&+ \big(\beta_1\beta_2\alpha_2\mu_2^1 - \beta_1\beta_2\alpha_1\mu_1^1 + \beta_1\beta_2\nu^1 - \beta_2\alpha_1\mathrm{L}_A\left(\beta_1\right) - \beta_2\mathrm{L}_{B_2}\left(\beta_1\right)\big) B_1 \\
&+ \big(\beta_1\beta_2\alpha_2\mu_2^2 - \beta_1\beta_2\alpha_1\mu_1^2 + \beta_1\beta_2\nu^2 + \beta_1\alpha_2\mathrm{L}_A\left(\beta_2\right) + \beta_1\mathrm{L}_{B_1}\left(\beta_2\right)\big) B_2.
\end{aligned}$$

Thus,

$$
\begin{aligned}
\tilde{\nu}^1 &= \frac{1}{\beta_1}\left(\beta_1\beta_2\alpha_2\mu_2^1 - \beta_1\beta_2\alpha_1\mu_1^1 + \beta_1\beta_2\nu^1 - \beta_2\alpha_1\mathrm{L}_A\left(\beta_1\right) - \beta_2\mathrm{L}_{B_2}\left(\beta_1\right)\right),\\
&= \beta_2\left(\alpha_2\mu_2^1 - \alpha_1\mu_1^1 + \nu^1 - \alpha_1\mathrm{L}_A\left(\ln(\beta_1)\right) - \mathrm{L}_{B_2}\left(\ln(\beta_1)\right)\right),\\
\tilde{\nu}^2 &= \frac{1}{\beta_2}\left(\beta_1\beta_2\alpha_2\mu_2^2 - \beta_1\beta_2\alpha_1\mu_1^2 + \beta_1\beta_2\nu^2 + \beta_1\alpha_2\mathrm{L}_A\left(\beta_2\right) + \beta_1\mathrm{L}_{B_1}\left(\beta_2\right)\right),\\
&= \beta_1\left(\alpha_2\mu_2^2 - \alpha_1\mu_1^2 + \nu^2 + \alpha_2\mathrm{L}_A\left(\ln(\beta_2)\right) + \mathrm{L}_{B_1}\left(\ln(\beta_2)\right)\right),\\
\tilde{\nu}^0 &= \frac{1}{\beta_1\beta_2}\left(\beta_1\beta_2\alpha_2\mu_2^0 - \beta_1\beta_2\alpha_1\mu_1^0 + \beta_1\beta_2\nu^0 + \beta_1\alpha_2\mathrm{L}_A\left(\beta_2\alpha_1\right)\right.\\
&\left.-\beta_2\alpha_1\mathrm{L}_A\left(\beta_1\alpha_2\right) - \beta_2\mathrm{L}_{B_2}\left(\beta_1\alpha_2\right) + \beta_1\mathrm{L}_{B_1}\left(\beta_2\alpha_1\right)\right)\\
&- \frac{\alpha_2}{\beta_1\beta_2}\left(\beta_1\beta_2\alpha_2\mu_2^1 - \beta_1\beta_2\alpha_1\mu_1^1 + \beta_1\beta_2\nu^1 - \beta_2\alpha_1\mathrm{L}_A\left(\beta_1\right) - \beta_2\mathrm{L}_{B_2}\left(\beta_1\right)\right)\\
&- \frac{\alpha_1}{\beta_1\beta_2}\left(\beta_1\beta_2\alpha_2\mu_2^2 - \beta_1\beta_2\alpha_1\mu_1^2 + \beta_1\beta_2\nu^2 + \beta_1\alpha_2\mathrm{L}_A\left(\beta_2\right) + \beta_1\mathrm{L}_{B_1}\left(\beta_2\right)\right),\\
&= \nu^0 - \alpha_2\nu^1 - \alpha_1\nu^2 + (\alpha_1)^2\mu_1^2 - (\alpha_2)^2\mu_2^1 + \alpha_1\alpha_2(\mu_1^1 - \mu_2^2)\\
&\quad + \alpha_2\mu_2^0 - \alpha_1\mu_1^0 + \alpha_2\mathrm{L}_A\left(\alpha_1\right) - \alpha_1\mathrm{L}_A\left(\alpha_2\right) + \mathrm{L}_{B_1}\left(\alpha_1\right) - \mathrm{L}_{B_2}\left(\alpha_2\right).
\end{aligned}
$$

We now compute the transformation of the structure functions γ^k.

$$
\begin{aligned}
\tilde{\gamma}^0\tilde{A} + \tilde{\gamma}^1\tilde{B}_1 + \tilde{\gamma}^2\tilde{B}_2 &= \tilde{C} = C + \alpha_1\alpha_2 A + \alpha_1 B_1 + \alpha_2 B_2,\\
&= \left(\gamma^0 + \alpha_1\alpha_2\right)A + \left(\gamma^1 + \alpha_1\right)B_1 + \left(\gamma^2 + \alpha_2\right)B_2,\\
&= \frac{1}{\beta_1\beta_2}\left(\gamma^0 + \alpha_1\alpha_2 - \alpha_2\left(\gamma^1 + \alpha_1\right) - \alpha_1\left(\gamma^2 + \alpha_2\right)\right)\tilde{A}\\
&+ \frac{1}{\beta_1}\left(\gamma^1 + \alpha_1\right)\tilde{B}_1 + \frac{1}{\beta_2}\left(\gamma^2 + \alpha_2\right)\tilde{B}_2,\\
&= \frac{1}{\beta_1\beta_2}\left(\gamma^0 - \alpha_2\gamma^1 - \alpha_1\gamma^2 - \alpha_1\alpha_2\right)\tilde{A} + \frac{1}{\beta_1}\left(\gamma^1 + \alpha_1\right)\tilde{B}_1 + \frac{1}{\beta_2}\left(\gamma^2 + \alpha_2\right)\tilde{B}_2.
\end{aligned}
$$

And finally relation (4.34) follows from the same application of the Jacobi identity as for p-elliptic systems (see relation (4.14), and Appendix 4.C).

Chapter 5

Control systems with paraboloid nonholonomic constraints in any dimension

This chapter is dedicated to generalising to any dimension the results of Chapter 4; recall that some general constructions and considerations are in Chapter 3. Our purpose is to propose a characterisation and a classification of (p, q)-paraboloid hypersurfaces $\mathcal{S}_Q$ of the tangent bundle $T\mathcal{X}$ of a smooth n-dimensional manifold $\mathcal{X}$, with $n \geq 3$. In a suitable coordinate system $x = (z, y)$, with $y = (y_1, \ldots, y_{n-1})$, they are given by

$$\mathcal{S}_Q = \{(x, \dot{x}) \in T\mathcal{X},\ \dot{z} = \dot{y}^t Q(x)\dot{y} + b(x)\dot{y} + c(x)\},$$

where Q is a smooth symmetric $(n-1) \times (n-1)$-matrix of full rank with constant signature (p, q), $b = (b_1, \ldots, b_{n-1})$ is a smooth covector, and c is a smooth scalar function. The problem of their characterisation is replaced by that of characterising their second prolongation $\Sigma_{\mathcal{S}_Q}$ among the class of control-affine systems $\Sigma = (f, g)$ with state space $\mathcal{M}$, a manifold of dimension $2n - 1$, and $m = n - 1$ controls. Thus, throughout this chapter we suppose $m \geq 2$.

Lemma 3.1 of Chapter 3 shows that the class of second prolongations of (p, q)-paraboloid submanifolds given by control-affine systems of the form $\Sigma_{\mathcal{S}_Q}$ is the same as the one given by the so-called (p, q)-paraboloid systems, shortly (p, q)-systems, of the form

$$\Sigma_{p,q} : \left\{ \begin{array}{ll} \dot{x} & = A(x)w^t \mathtt{I}_{p,q} w + \sum_{i=1}^m B_i(x)w_i + C(x) \\ \dot{w} & = u \end{array} \right. , \quad (x, w) \in \mathcal{M} \text{ and } u \in \mathbb{R}^m,$$

where the constant matrix $\mathtt{I}_{p,q}$ is defined by $\mathtt{I}_{p,q} = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}$ and its elements are denoted by $\mathtt{I}^i_j$, in particular we have $\mathtt{I}^i_j = 0$ if $i \neq j$. Therefore, in order to propose a characterisation of (p, q)-paraboloid submanifolds, we will study the feedback equivalence of control-affine systems $\Sigma = (f, g)$ of the form

$$\Sigma : \dot{\xi} = f(\xi) + \sum_{i=1}^m u_i g_i(\xi),$$

with state $\xi \in \mathcal{M}$, a $(2m+1)$-dimensional manifold, and controls $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ with a (p, q)-paraboloid system $\Sigma_{p,q}$. Recall that we attach to $\Sigma = (f, g)$ the following two distributions

$$\mathcal{D}^0 = \text{span}\,\{g_1, \dots g_m\} \quad \text{and} \quad \mathcal{D}^1 = \text{span}\,\{g_1, \dots, g_m, \text{ad}_f g_1, \dots, \text{ad}_f g_m\}\,.$$

Moreover, we recall the following general assumptions that are necessary for our characterisation (see Chapter 3 and, in particular, Lemma 3.2 and Proposition 3.1 for the definition and some properties of the application Ω_ω):

(A0) The number of controls is $m \geq 2$,

(A1) The distribution $\mathcal{D}^0$ is involutive and has constant rank m,

(A2) The distribution $\mathcal{D}^1$ has constant rank $2m$,

(A3) $\text{sgn}\,(\Omega_\omega) = (p, q)$ is constant and satisfies $p + q = m$.

We add assumption (A0) to emphasize that our approach applies to the case $m \geq 2$ only. We explain after Theorem 5.3 why the case $m = 1$ has to be treated (and were treated) separately.

Our second purpose is to propose a classification of (p, q)-paraboloid submanifolds $\mathcal{S}_Q$. This is done via a classification of the orbits under feedback transformations of their first prolongations, expressed as

$$\Xi_{p,q} \,:\, \dot{x} = A(x) w^t \mathtt{I}_{p,q} w + \sum_{i=1}^{m} B_i(x) w_i + C(x), \quad x \in \mathcal{X} \text{ and } w \in \mathbb{R}^m.$$

Recall that the chosen nomenclature for the normal and canonical forms is given in Table 1 of Chapter 1.

This chapter is organised as follows. In the next section, we will fully characterise the class of (p, q)-paraboloid systems $\Sigma_{p,q}$. Our conditions are necessary and sufficient, and can explicitly be computed via structure functions attached to control-affine systems Σ (see Theorem 5.3). Afterwards, we will treat the problem of classifying (p, q)-paraboloid submanifolds via the the study of some orbits under feedback transformations of (p, q)-systems $\Xi_{p,q}$ (see Theorem 5.6). Our classification is expressed with several normal and canonical forms of (p, q)-paraboloid systems, that yields analogous normal and canonical forms for (p, q)-paraboloid submanifolds (see Corollary 5.2).

1 Characterisation of (p, q)-paraboloid systems

In this section, we will characterise the class of (p, q)-paraboloid systems represented by control-affine systems of the following form

$$\Sigma_{p,q} \,:\, \begin{cases} \dot{x} &= A(x) w^t \mathtt{I}_{p,q} w + \sum_{i=1}^{m} B_i(x) w_i + C(x) \\ \dot{w} &= u \end{cases},$$

where $(x, w) \in \mathcal{M}$ is the state and $u = (u_1, \dots, u_m)^t \in \mathbb{R}^m$ is the control. Moreover, A, $B_1, \dots, B_m$, and C are smooth vector fields satisfying $A \wedge B_1 \wedge \dots \wedge B_m \neq 0$ on the $(m+1)$-dimensional quotient manifold $\mathcal{M}/\mathcal{D}^0$, which locally is well defined because $\mathcal{D}^0 = \text{span}\left\{\frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial w_m}\right\}$ is involutive and of constant rank m. Our characterisation given in Theorem 5.3 can explicitly be tested on any control system

$\Sigma = (f, g)$, of proper dimensions, by means of algebraic and differential relations between well-defined structure functions.

Consider a control-affine system $\Sigma = (f, g)$ with state space $\mathcal{M}$, a smooth $(2m+1)$-dimensional manifold, and with m controls.

Definition 5.1 (Weak quadratic frame)**.** We say that the m-tuple $g = (g_1, \ldots, g_m)$ of a control-affine system $\Sigma = (f, g)$ is a *weak quadratic frame* of the distribution $\mathcal{D}^0 = \operatorname{span}\{g_1, \ldots, g_m\}$, shortly a WQF, if there exists a smooth vector field $Z \notin \mathcal{D}^1$ such that

$$[g_i, \mathrm{ad}_f g_j] = \mathtt{I}_j^i \, Z \mod \mathcal{D}^1.$$

In particular, our definition implies that $[g_i, \mathrm{ad}_f g_j] \in \mathcal{D}^1$ for all $i \neq j$. Although the vector field Z is not unique, it is unique modulo $\mathcal{D}^1$ and therefore we will denote a weak quadratic frame by the pair (g, Z). Recall the map Ω_ω defined in Chapter 3. For a weak quadratic frame, we have $\Omega_\omega = \eta \mathtt{I}_{p,q}$ for some smooth function $\eta \neq 0$ (depending on the choice of ω); actually, by assumption (A3), if $p \neq q$ then $\eta > 0$. Assume that (g, Z) is a WQF, then we can define structure functions $\mu_{i,j}^k$, for $i, j, k = 1, \ldots, m$, by

$$[g_i, \mathrm{ad}_f g_j] = \mathtt{I}_j^i \, Z + \sum_{k=1}^{m} \mu_{i,j}^k \, \mathrm{ad}_f g_k \mod \mathcal{D}^0.$$

It will be useful to denote by μ_i the matrix whose elements are $\mu_{i,j}^k$. We will also use the structure functions $\nu_{i,j}^k$ uniquely defined by

$$[g_i, g_j] = \sum_{k=1}^{m} \nu_{i,j}^k \, g_k.$$

By a direct application of the Jacobi identity to $[g_i, \mathrm{ad}_f g_j]$, we immediately obtain the following relation between the structure functions:

$$\mu_{i,j}^k - \nu_{i,j}^k - \mu_{j,i}^k = 0. \tag{5.1}$$

By a straightforward calculation, we deduce that the fact of being a WQF and the associated structure functions $\mu_{i,j}^k$ do not depend on feedback transformations of the form $f \mapsto f + \sum_{k=1}^{m} \alpha_k g_k$.

Remark (On the uniqueness of Z). As we already mentioned, for a weak quadratic frame (g, Z), the vector field Z is not unique; we can change it (modulo $\mathcal{D}^0$) by $Z \mapsto \tilde{Z} = Z + \sum_{k=1}^{m} \upsilon^k \, \mathrm{ad}_f g_k$, for some smooth functions υ^k. Under that transformation, the structure functions $\mu_{i,j}^k$ associated with the WQF (g, Z) are mapped into the structure functions $\tilde{\mu}_{i,j}^k$ of $(g, \tilde{Z})$ given by $\tilde{\mu}_{i,j}^k = \mu_{i,j}^k - \mathtt{I}_j^i \upsilon^k$, implying that

$$\tilde{\mu}_{i,j}^k = \mu_{i,j}^k \quad \text{if} \quad i \neq j, \quad \text{and} \quad \tilde{\mu}_{i,i}^k = \mu_{i,i}^k - \mathtt{I}_i^i \upsilon^k \quad \text{otherwise}.$$

With our matrix notation, $\mu_i = (\mu_{i,j}^k)$, that relation can be summarise as:

$$\tilde{\mu}_i = \begin{pmatrix} \mu_{i,1}^1 & & & & \\ & \ddots & & * & \\ & & \mu_{i,i}^i & & \\ & * & & \ddots & \\ & & & & \mu_{i,m}^m \end{pmatrix} - \mathtt{I}_i^i \begin{pmatrix} 0 & & \upsilon^1 & & \\ & \ddots & \vdots & 0 & \\ & & \upsilon^i & & \\ & 0 & \vdots & \ddots & \\ & & \upsilon^m & & 0 \end{pmatrix}.$$

◆

The following proposition shows that a weak quadratic frame always exists (under the general assumptions (A1), (A2), and (A3)), characterises the feedback transformations that preserve a WQF, and describes how the structure functions of two equivalent WQF are related. Recall that $\beta(\cdot) \in GO(p,q)$, the conformal group, i.e. $\beta^t \mathtt{I}_{p,q}\beta = \lambda \mathtt{I}_{p,q}$, where the function λ is called the associated multiplier.

Proposition 5.1 (Existence and properties of weak quadratic frames)**.**

(i) *Under assumptions (A1), (A2), and (A3) there exists a weak quadratic frame (g, Z).*

(ii) *If (g, Z) is a weak quadratic frame, then $(\tilde{g}, \tilde{Z})$, with $\tilde{g} = g\beta$, is also a weak quadratic frame if and only if $\beta \in C^\infty(\mathcal{M}, GO(p,q))$, i.e. $\beta^t I_{p,q}\beta = \lambda I_{p,q}$ for some smooth associated multiplier $\lambda(\cdot) \neq 0$.*

(iii) *Consider two weak quadratic frames (g, Z) and $(\tilde{g}, \tilde{Z})$, with structure functions $\mu^k_{i,j}$ and $\tilde{\mu}^k_{i,j}$, respectively. Assume that they are related by the feedback $\tilde{g} = g\beta$, with $\beta(\cdot) \in GO(p,q)$ and associated multiplier λ. Then, $\tilde{Z} = \lambda\left(Z + \sum_{k=1}^m \upsilon^k \mathrm{ad}_f g_k\right) \mod \mathcal{D}^0$, for some smooth functions υ^k, and the structure functions are related by*

$$\lambda I^i_j \upsilon^k + \beta^k_s \tilde{\mu}^s_{i,j} = \beta^s_i \mu^k_{s,r}\beta^r_j + \beta^s_i \mathrm{L}_{g_s}\left(\beta^k_j\right). \tag{5.2}$$

The Einstein summation convention is used over indices r and s.

(iv) *Under assumptions (A1), (A2), and (A3), an m-tuple $g = (g_1, \ldots, g_m)$ is a weak quadratic frame of $\mathcal{D}^0$ if and only if*

$$\begin{aligned} [g_i, \mathrm{ad}_f g_i]\, I^i_i - [g_j, \mathrm{ad}_f g_j]\, I^j_j &= 0 \mod \mathcal{D}^1 \quad \textit{for all} \quad i,j = 1,\ldots,m, \\ [g_i, \mathrm{ad}_f g_j] &= 0 \mod \mathcal{D}^1 \quad \textit{if} \quad i \neq j. \end{aligned} \tag{5.3}$$

In relation (5.2), the functions υ^k correspond to the different choices of the vector field Z. Notice that statement *(iv)* of the above proposition gives an equivalent definition of WQF which does not depend on the choice of a vector field Z; compare that expression with the definitions of weak orthonormal and isotropic frames (see Definitions 4.2 and 4.8 in Chapter 4). Characterisation (5.3) gives a computational way to check if a given frame g of $\mathcal{D}^0$ is, actually, a weak quadratic one. A drawback of that characterisation is that it is harder to identify the structure functions $\mu^k_{i,j}$ in a systematic manner.

Proof.

(i) Suppose that assumptions (A1), (A2), and (A3) hold, for a fixed ω. It follows from Proposition 3.1 of Chapter 3 that there exists a frame $\tilde{g} = g\beta$ of $\mathcal{D}^0$ such that $\tilde{\Omega}_\omega = \mathtt{I}_{p,q}$. Thus, the pair $(\tilde{g}, Z)$ is a weak quadratic frame for any vector field Z satisfying $\omega(Z) = 1$.

(ii) Let (g, Z) and $(\tilde{g}, \tilde{Z})$, with $\tilde{g} = g\beta$, be two weak quadratic frames. By a direct calculation, modulo $\mathcal{D}^1$, we obtain that

$$\mathtt{I}^i_j \tilde{Z} = [\tilde{g}_i, \mathrm{ad}_f \tilde{g}_j] = \left[g_s \beta^s_i, \mathrm{ad}_f g_r\, \beta^r_j\right] = \beta^s_i \mathtt{I}^s_r \beta^r_j Z,$$

where summation over the indices r and s is used. Clearly, modulo the distribution $\mathcal{D}^1$, we have $\tilde{Z} = \lambda Z$ for some smooth function λ satisfying $\lambda(\cdot) \neq 0$. Therefore, for all i, j we obtain $\lambda \mathtt{I}^i_j = \beta^s_i \mathtt{I}^s_r \beta^r_j$; that is $\lambda \mathtt{I}_{p,q} = \beta^t \mathtt{I}_{p,q}\beta$. The converse is immediate.

(iii) Let (g, Z) and $(\tilde{g}, \tilde{Z})$ be two weak quadratic frames related by $\tilde{g} = g\beta$, with $\beta \in GO(p,q)$ and associated multiplier λ. Then, by a direct computation modulo the distribution $\mathcal{D}^0$, the structure functions $\mu^k_{i,j}$ of (g, Z) and $\tilde{\mu}^k_{i,j}$ of $(\tilde{g}, \tilde{Z})$ are related by

$$[\tilde{g}_i, \mathrm{ad}_f \tilde{g}_j] = \left[g_s\beta^s_i, \mathrm{ad}_f \left(g_r\beta^r_j\right)\right] = \beta^s_i \left[g_s, \mathrm{ad}_f g_r\right] \beta^r_j + \beta^s_i \mathrm{L}_{g_s}\left(\beta^r_j\right) \mathrm{ad}_f g_r,$$
$$\mathtt{I}^i_j \tilde{Z} + \mathrm{ad}_f g_k\, \beta^k_s \tilde{\mu}^s_{i,j} = \beta^s_i \mathtt{I}^r_s \beta^r_j Z + \left(\beta^s_i \mu^k_{s,r}\beta^r_j + \beta^s_i \mathrm{L}_{g_s}\left(\beta^k_j\right)\right) \mathrm{ad}_f g_k.$$

Applying any differential one-form ω such that $\mathrm{ann}\left(\mathcal{D}^1\right) = \mathrm{span}\left\{\omega\right\}$ to the last equation, we get $\mathtt{I}^i_j\omega(\tilde{Z}) = \lambda \mathtt{I}^i_j \omega(Z)$, where λ is defined by the relation $\beta^t \mathtt{I}_{p,q}\beta = \lambda \mathtt{I}_{p,q}$. Hence, $\omega(\tilde{Z} - \lambda Z) = 0$, that is $\tilde{Z} - \lambda Z = 0 \mod \mathcal{D}^1$. Therefore, there exists smooth functions υ^k such that $\tilde{Z} = \lambda\left(Z + \sum_{k=1}^m \upsilon^k \mathrm{ad}_f g_k\right)$ mod $\mathcal{D}^0$. Implying that $\lambda \mathtt{I}^i_j \upsilon^k + \beta^k_s \tilde{\mu}^s_{i,j} = \beta^s_i \mu^k_{s,r}\beta^r_j + \beta^s_i \mathrm{L}_{g_s}\left(\beta^k_j\right)$.

(iv) If (g, Z) is a weak quadratic frame, then clearly,

$$[g_i, \mathrm{ad}_f g_j] = 0 \mod \mathcal{D}^1 \quad \text{for} \quad i \neq j,$$
$$\text{and} \quad [g_i, \mathrm{ad}_f g_i]\, \mathtt{I}^i_i - [g_j, \mathrm{ad}_f g_j]\, \mathtt{I}^j_j = \mathtt{I}^i_i Z \mathtt{I}^i_i - \mathtt{I}^j_j Z \mathtt{I}^j_j = 0 \mod \mathcal{D}^1,$$

recall that $(\mathtt{I}^i_i)^2 = 1$ for all $1 \leq i \leq m$. Conversely, assume that an m-tuple $g = (g_1, \ldots, g_m)$ satisfies (5.3). Obviously, for $i \neq j$ we have $[g_i, \mathrm{ad}_f g_j] = \mathtt{I}^i_j Z$ mod $\mathcal{D}^1$ (for some non-zero vector field Z). Next, observe that assumption (A3) implies that $[g_i, \mathrm{ad}_f g_i] \notin \mathcal{D}^1$ for all $1 \leq i \leq m$. Recall that $\mathtt{I}^1_1 = 1$, so we set $Z = [g_1, \mathrm{ad}_f g_1]$ and thus we deduce that $[g_i, \mathrm{ad}_f g_i] = \mathtt{I}^i_i Z \mod \mathcal{D}^1$.

■

We now reinforce the notion of weak quadratic frames which will turn out to be the key of our characterisation of (p,q)-paraboloid systems.

Definition 5.2 (Strong quadratic frame)**.** We say that the m-tuple $g = (g_1, \ldots, g_m)$ of a control-affine system $\Sigma = (f, g)$ is a *strong quadratic frame* of the distribution $\mathcal{D}^0 = \mathrm{span}\left\{g_1, \ldots, g_m\right\}$, shortly a SQF, if there exists a smooth vector field $Z \notin \mathcal{D}^1$ such that

$$[g_i, \mathrm{ad}_f g_j] = \mathtt{I}^i_j Z \mod \mathcal{D}^0.$$

Clearly, strong quadratic frames form a subclass of weak quadratic frames whose structure functions are given by $\mu^k_{i,j} = 0$, for all $1 \leq i, j, k \leq m$. For a SQF, the vector field Z is uniquely determined modulo $\mathcal{D}^0$, indeed, transformations of the form $Z \mapsto \tilde{Z} = Z + \sum_{k=1}^m \upsilon^k \mathrm{ad}_f g_k$ do not preserve the property of being a strong quadratic frame. The following proposition details some properties of SQFs.

Proposition 5.2 (Properties of strong quadratic frames)**.**

(i) Any (p,q)-paraboloid system $\Sigma_{p,q}$ possesses a strong quadratic frame;

(ii) If $(\tilde{g}, \tilde{Z})$ is a strong quadratic frame, then (g, Z) defined by $\tilde{g} = g\beta$, with $\beta \in C^\infty(\mathcal{M}, GO(p,q))$, is a weak quadratic frame with $\tilde{Z} = \lambda\left(Z + \sum_{k=1}^m \upsilon^k \mathrm{ad}_f g_k\right)$ mod $\mathcal{D}^0$ and whose structure functions $\mu^k_{i,j}$ satisfy

$$\lambda I^i_j \upsilon^k = \beta^s_i \mu^k_{s,r} \beta^r_j + \beta^s_i \mathrm{L}_{g_s}\left(\beta^k_j\right). \tag{5.4}$$

(iii) *If (g, Z) is a strong quadratic frame, then for all $1 \leq i, j \leq m$ we have $[g_i, g_j] = 0$.*

(iv) *Under assumptions (A1), (A2), and (A3), the m-tuple $g = (g_1, \ldots, g_m)$ is a strong quadratic frame of $\mathcal{D}^0$ if and only if*

$$
\begin{aligned}
[g_i, \mathrm{ad}_f g_i]\, I^i_i - [g_j, \mathrm{ad}_f g_j]\, I^j_j &= 0 \mod \mathcal{D}^0 \quad \textit{for all} \quad i, j = 1, \ldots, m, \\
\textit{and} \quad [g_i, \mathrm{ad}_f g_j] &= 0 \mod \mathcal{D}^0 \quad \textit{if} \quad i \neq j.
\end{aligned}
\tag{5.5}
$$

Statement *(iv)* of the above proposition gives an alternative definition of a strong quadratic frame which does not depend on the choice of Z (compare with the definitions of strong orthonormal and isotropic frames; see Definitions 4.3 and 4.9 of Chapter 4). That alternative definition will be useful in the proof of Theorem 5.3.

Proof.

(i) In coordinates (x, w), recall that any (p, q)-system $\Sigma_{p,q} = (f, g)$ is given by the vector fields $g_i = \frac{\partial}{\partial w_i}$, for $1 \leq i \leq m$, and $f = A(x) w^t \mathtt{I}_{p,q} w + B(x) w + C(x) \mod \mathcal{D}^0$. It is a straightforward computation to show that (g, Z), with $Z = -2A\frac{\partial}{\partial x}$, is a strong quadratic frame. Indeed, on one hand, since $A \wedge B_1 \wedge \ldots \wedge B_m \neq 0$ we have $A\frac{\partial}{\partial x} \notin \mathcal{D}^1 = \mathcal{D}^0 + \mathrm{span}\left\{(2A\mathtt{I}^i_i w_i + B_i)\frac{\partial}{\partial x},\ 1 \leq i \leq m\right\}$, and, in the other hand, we have $[g_i, \mathrm{ad}_f g_j] = -2A\mathtt{I}^i_j \frac{\partial}{\partial x} \mod \mathcal{D}^0$.

(ii) Assume that $(\tilde{g}, \tilde{Z})$ is a strong quadratic frame and let $g = \tilde{g}\beta^{-1}$, with $\beta \in C^\infty(\mathcal{M}, GO(p, q))$, then clearly (g, Z) is a weak quadratic frame by Proposition 5.1 *(ii)*, and formula (5.4) immediately derives from the application of relation (5.2) with $\tilde{\mu}^k_{i,j} = 0$.

(iii) Recall that the structure functions $\nu^k_{i,j}$ are defined through $[g_i, g_j] = \sum_{k=1}^m \nu^k_{i,j} g_k$. Suppose that g is a strong quadratic frame, then using relation (5.1) we obtain

$$
\nu^k_{i,j} = \mu^k_{i,j} - \mu^k_{j,i} = 0,
$$

implying $[g_i, g_j] = 0$.

(iv) If (g, Z) is a strong quadratic frame then, we clearly have

$$
\begin{aligned}
[g_i, \mathrm{ad}_f g_j] &= 0 \mod \mathcal{D}^0 \quad \text{for} \quad i \neq j, \\
\text{and} \quad [g_i, \mathrm{ad}_f g_i]\, \mathtt{I}^i_i - [g_j, \mathrm{ad}_f g_j]\, \mathtt{I}^j_j &= \mathtt{I}^i_i Z \mathtt{I}^i_i - \mathtt{I}^j_j Z \mathtt{I}^j_j = 0 \mod \mathcal{D}^0.
\end{aligned}
$$

Conversely, if an m-tuple $g = (g_1, \ldots, g_m)$ satisfies (5.5), then obviously for $i \neq j$ we have $[g_i, \mathrm{ad}_f g_j] = \mathtt{I}^i_j Z$ (for some non-zero vector field Z). Next, observe that assumption (A3) implies that $[g_i, \mathrm{ad}_f g_i] \notin \mathcal{D}^1$, for all $1 \leq i \leq m$. Recall that $\mathtt{I}^1_1 = 1$, so we set $Z = [g_1, \mathrm{ad}_f g_1]$ and we deduce that $[g_i, \mathrm{ad}_f g_i] = \mathtt{I}^i_i Z \mod \mathcal{D}^0$, for all $1 \leq i \leq m$.

∎

We have now set everything for the characterisation of (p, q)-paraboloid systems, in terms of algebraic and differential relations between the structure functions $\mu^k_{i,j}$. Statement *(i)* of the above proposition asserts that any $\Sigma_{p,q}$ possesses a strong quadratic frame thus the existence of a SQF is a necessary condition for the equivalence of a control-affine system Σ with $\Sigma_{p,q}$. In the following paragraph, we will exploit relation (5.4) to derive necessary conditions on the structure functions $\mu_{i,j}^k$ of a WQF that is equivalent to a SQF. Next we will show that those conditions are also sufficient.

Consider a WQF (g, Z) and assume that it is equivalent to a SQF, implying that there exists a feedback $\beta \in C^\infty(\mathcal{M}, GO(p,q))$, with associated multiplier λ, and smooth functions υ^k such that relation (5.4) hold. First, using the relation $\lambda \mathtt{I}_j^i = \beta_i^s \mathtt{I}_r^s \beta_j^r$, we rewrite equation (5.4) as

$$\beta_i^s \left(\mu_{s,r}^k - \mathtt{I}_r^s \upsilon^k\right) \beta_j^r + \beta_i^s \mathrm{L}_{g_s}\left(\beta_j^k\right) = 0. \tag{5.4'}$$

From that relation, we deduce that $\left(\mu_{s,r}^k - \mathtt{I}_r^s \upsilon^k\right) \beta_j^r + \mathrm{L}_{g_s}\left(\beta_j^k\right) = 0$, for all $1 \leq s, k, j \leq m$. We denote by $\tilde{\mu}_s$, for all $1 \leq s \leq m$, the matrix whose elements are $\tilde{\mu}_{s,r}^k = \mu_{s,r}^k - \mathtt{I}_r^s \upsilon^k$. Observe that $\tilde{\mu}_{s,r}^k$ are the structure function of the WQF $(g, \tilde{Z})$, with $\tilde{Z} = Z + \sum_{k=1}^m \upsilon^k \mathrm{ad}_f g_k$. Using that matrix notation, we conclude that $\tilde{\mu}_s \beta + \mathrm{L}_{g_s}(\beta) = 0$, for all $s = 1, \ldots, m$. Thus, by Lemma 3.3 of Chapter 3, we conclude that

$$\forall\, s = 1, \ldots, m, \quad \tilde{\mu}_s + \frac{1}{2\lambda} \mathrm{L}_{g_s}(\lambda)\, \mathrm{Id}_m \in \mathrm{Lie}\,(O(p,q))\,. \tag{5.6}$$

Since each element of $\mathrm{Lie}\,(O(p,q))$ has its diagonal identically equal to zero, we deduce that if $\tilde{\mu}_s$ fulfils the above inclusion, then the following two conditions hold for all $1 \leq s \leq m$:

$$\forall\, i, j = 1, \ldots, m, \quad \tilde{\mu}_{s,i}^i = \tilde{\mu}_{s,j}^j, \tag{5.7a}$$

$$\tilde{\mu}_s^\triangle := \tilde{\mu}_s - \mathrm{diag}\,(\tilde{\mu}_s) \in \mathrm{Lie}\,(O(p,q))\,. \tag{5.7b}$$

When condition (5.7a) is fulfilled, we denote $\tilde{\sigma}_s := (\tilde{\mu}_s)_1^1$ and thus we have $\tilde{\mu}_s^\triangle := \tilde{\mu}_s - \tilde{\sigma}_s \mathrm{Id}_m$. The matrices of structure functions $\tilde{\mu}_s$ are related with the feedback transformations $\beta \in C^\infty(\mathcal{M}, GO(p,q))$ by the following systems of linear first order partial differential equations for the unknowns λ and β

$$\forall\, i = 1, \ldots, m \qquad \frac{1}{2\lambda} \mathrm{L}_{g_i}(\lambda) = -\tilde{\sigma}_i \quad \text{and} \quad \mathrm{L}_{g_i}(\beta) = -\tilde{\mu}_i \beta. \tag{5.8}$$

The integrability conditions of those system are given by, recall the structure functions $\nu_{i,j}^k$ defined by $[g_i, g_j] = \sum_{k=1}^m \nu_{i,j}^k g_k$,

$$\mathrm{L}_{g_i}(\tilde{\sigma}_j) - \mathrm{L}_{g_j}(\tilde{\sigma}_i) = \sum_{k=1}^m \tilde{\sigma}_k\, \nu_{i,j}^k, \tag{5.9a}$$

$$\text{and} \quad \mathrm{L}_{g_i}(\tilde{\mu}_j) - \mathrm{L}_{g_j}(\tilde{\mu}_i) + \tilde{\mu}_i \tilde{\mu}_j - \tilde{\mu}_j \tilde{\mu}_i = \sum_{k=1}^m \tilde{\mu}_k\, \nu_{i,j}^k,$$

$$\text{implying} \quad \mathrm{L}_{g_i}\left(\tilde{\mu}_j^\triangle\right) - \mathrm{L}_{g_j}\left(\tilde{\mu}_i^\triangle\right) + \tilde{\mu}_i^\triangle \tilde{\mu}_j^\triangle - \tilde{\mu}_j^\triangle \tilde{\mu}_i^\triangle = \sum_{k=1}^m \tilde{\mu}_k^\triangle\, \nu_{i,j}^k. \tag{5.9b}$$

Conditions (5.7a), (5.7b), (5.9a), and (5.9b) are the core of our characterisation of strong quadratic frames. Those conditions, however, are not expressed directly in terms of the structure functions $\mu_{i,j}^k$ of a given WQF, but on a modified version of them; actually, for a WQF for which the vector field Z has been suitably chosen. In the following, we explain how different choices of Z twist condition (5.7a) and, as a consequence, it will provide us a way of constructing that «good» Z.

Recall that the matrices of structure functions μ_s and of $\tilde{\mu}_s$ (corresponding to two WQFs (g, Z) and $(g, \tilde{Z})$, respectively) are related by $\tilde{\mu}^i_{s,i} = \mu^i_{s,i} - \mathtt{I}^s_i \upsilon^i$ and thus their diagonal $\operatorname{diag}(\mu_s)$ and $\operatorname{diag}(\tilde{\mu}_s)$ differ at most in one location. Hence, we deduce that if $\tilde{\mu}_s$ satisfy (5.7a), then, for all indices $1 \le s \le m$, it holds

(5.7a') $$\forall\, 1 \le i, j \ne s \le m, \quad (\mu_s)^i_i = (\mu_s)^j_j,$$
$$\text{and} \quad \exists\, \upsilon^s \in C^\infty(\mathcal{M}),\ \forall\, i \ne s \quad (\mu_s)^s_s - (\mu_s)^i_i = \mathtt{I}^s_s \upsilon^s.$$

Clearly, if the first part of that condition is satisfied, then the existence of the function υ^k is automatic, as we describe it below; we include that second part so that the function υ^k explicitly appear in it. If that condition holds, then it provides a way to identify the functions υ^k. Indeed, in that case, we have $(\mu_s)^s_s - (\mu_s)^i_i = \mathtt{I}^s_s \upsilon^s$ for all $1 \le i \ne s \le m$. Therefore, if (g, Z) is a WQF satisfying (5.7a'), then we can uniquely identify smooth function υ^k, using which we can transform the field Z into $\tilde{Z} = Z + \sum_{k=1}^m \upsilon^k \mathrm{ad}_f g_k$ and get structure function $\tilde{\mu}^k_{i,j}$ for which conditions (as (5.7b) and its consequences) can be tested.

The following theorem shows, first that conditions (5.7a), (5.7b), (5.9a), and (5.9b) are also sufficient for the existence of a SQF and, second, that the existence of a SQF fully characterises (p, q)-paraboloid systems.

Theorem 5.3 (Characterisation of (p, q)-paraboloid systems)**.** *Consider a control-affine system $\Sigma = (f, g)$ satisfying assumptions (A1), (A2), and (A3). Then, the following statements are locally equivalent,*

(S1) *Σ is feedback equivalent to $\Sigma_{p,q}$;*

(S2) *For any weak quadratic frame (g, Z), the structure functions $\mu^k_{i,j}$ satisfy (5.7a') and, additionally, the modified structure functions $\tilde{\mu}^k_{i,j}$ of $\left(g, Z + \sum_{k=1}^m \upsilon^k \mathrm{ad}_f g_k\right)$ satisfy (5.7b), and the systems of linear partial differential equations given by (5.8) have solutions (λ, β) fulfilling $\beta^t I_{p,q} \beta = \lambda I_{p,q}$;*

(S3) *For any weak quadratic frame (g, Z), the structure functions $\mu^k_{i,j}$ satisfy (5.7a') and, additionaly, the modified structure functions $\tilde{\mu}^k_{i,j}$ of $\left(g, Z + \sum_{k=1}^m \upsilon^k \mathrm{ad}_f g_k\right)$ satisfy (5.7b), (5.9a), and (5.9b);*

(S4) *There exists a strong quadratic frame of Σ;*

As announced, the equivalence of (S1) and (S4) shows that the existence of a strong quadratic frame is the key of the characterisation of the class of (p, q)-paraboloid systems. In order to test if a given control-affine system $\Sigma = (f, g)$ is feedback equivalent to $\Sigma_{p,q}$ we proceed as follow:

1) We test that assumptions (A1), (A2), and (A3) hold;
2) Using Proposition 3.1 of Chapter 3 we explicitly construct a weak quadratic frame (g, Z), to which we attach structure functions $\mu^k_{i,j}$;
3) We test condition (5.7a'), due to which we uniquely identify the functions υ^k;
4) We change Z by $Z \mapsto \tilde{Z} = Z + \sum_{k=1}^m \upsilon^k \mathrm{ad}_f g_k$ and calculate the modified structure functions $\tilde{\mu}^k_{i,j}$ of the weak quadratic frame $(g, \tilde{Z})$;

5) Finally, we test that (5.9a) and (5.9b) hold.

Proof. We will show (S1)⇒(S2)⇒(S3)⇒(S4)⇒(S1).
(S1)⇒(S2)⇒(S3). It is the analysis performed before the statement of the theorem.
(S3)⇒(S4). Assume that (g, Z) is a weak quadratic frame with structure functions $\mu_{i,j}^k$ satisfying (5.7a'), and that the structure functions $\tilde{\mu}_{i,j}^k$ of the modified WQF $\left(g, \tilde{Z} = Z + \sum_{k=1}^m v^k \mathrm{ad}_f g_k\right)$ satisfy conditions (5.7b), (5.9a), and (5.9b). The last two conditions imply that there exists smooth solutions $(\lambda, \beta^{\triangle})$ of the following two systems (see Corollary 3.1 of Chapter 3 for a proof of the existence of $\beta^{\triangle}$)

$$\frac{1}{2\lambda}\mathrm{L}_{g_i}(\lambda) = -\tilde{\sigma}_i \quad \text{and} \quad \mathrm{L}_{g_i}(\beta^{\triangle}) = -\tilde{\mu}_i^{\triangle}\beta^{\triangle}.$$

Without loss of generality, we can suppose that $\lambda > 0$ (because, if λ is a solution of the first system, so is $-\lambda$). Moreover, since $\mu_i^{\triangle} \in \mathrm{Lie}(O(p,q))$, due to Lemma 3.5 of Chapter 3 we have $\beta^{\triangle} \in O(p,q)$. Therefore, we construct the feedback $\beta = \sqrt{\lambda}\beta^{\triangle}$, which indeed belongs to $C^{\infty}(\mathcal{M}, GO(p,q))$, and by a direct application of relation (5.2) we deduce that the structure functions $\tilde{\tilde{\mu}}_{i,j}^k$ of the weak quadratic frame $(\tilde{\tilde{g}} = g\beta, \tilde{Z})$ satisfy

$$\begin{aligned}
\beta_s^k \tilde{\tilde{\mu}}_{i,j}^s &= \beta_i^s \left(\tilde{\mu}_s\beta + \mathrm{L}_{g_s}(\beta)\right)_j^k = \beta_i^s \left(\tilde{\mu}_s\sqrt{\lambda}\beta^{\triangle} + \mathrm{L}_{g_s}\left(\sqrt{\lambda}\beta^{\triangle}\right)\right)_j^k, \\
&= \beta_i^s \left(\tilde{\mu}_s^{\triangle}\sqrt{\lambda}\beta^{\triangle} + \tilde{\sigma}_s\sqrt{\lambda}\beta^{\triangle} + \frac{1}{2\sqrt{\lambda}}\mathrm{L}_{g_s}(\lambda)\,\beta^{\triangle} + \sqrt{\lambda}\mathrm{L}_{g_s}(\beta^{\triangle})\right)_j^k, \\
&= \beta_i^s \left(\sqrt{\lambda}\left[\tilde{\mu}_s^{\triangle}\beta^{\triangle} + \mathrm{L}_{g_s}(\beta)^{\triangle}\right] + \sqrt{\lambda}\beta^{\triangle}\left[\tilde{\sigma}_s + \frac{1}{2\lambda}\mathrm{L}_{g_s}(\lambda)\right]\beta^{\triangle}\right)_j^k = 0.
\end{aligned}$$

Therefore, $(\tilde{\tilde{g}}, \tilde{Z})$ is actually a strong quadratic frame.
(S4)⇒(S1). Let $g = (g_1, \ldots, g_m)$ be a strong quadratic frame, and recall that by statement *(iii)* of Proposition 5.2 the elements of this frame satisfy $[g_i, g_j] = 0$ for all $1 \le i, j \le m$. Therefore, we introduce coordinates $(x, w) = \phi(\xi)$ such that $\phi_* g_i = \frac{\partial}{\partial w_i}$, for $i = 1, \ldots, m$, and after applying a suitable feedback $f \mapsto f + \sum_{i=1}^m g_i\alpha^i$ the system Σ takes the form

$$\Sigma : \begin{cases} \dot{x} &= \mathsf{f}(x,w) \\ \dot{w} &= u \end{cases},$$

for which $g = \left(\frac{\partial}{\partial w_1}, \ldots, \frac{\partial}{\partial w_m}\right)$ is a strong quadratic frame. Using the characterisation of a strong quadratic frame given by relation (5.5), we have the following conditions on $\mathsf{f}(x,w)$

$$\frac{\partial^2 \mathsf{f}}{\partial w_i^2}\mathtt{I}_i^i - \frac{\partial^2 \mathsf{f}}{\partial w_j^2}\mathtt{I}_j^j = 0 \quad \text{and} \quad \frac{\partial^2 \mathsf{f}}{\partial w_i \partial w_j} = 0, \tag{5.10}$$

for all $1 \le i \ne j \le m$. The latter condition yields $\mathsf{f}(x,w) = \sum_{i=1}^m \mathsf{f}^i(x, w_i)$ which inserted in the former one implies $\frac{\partial^3 \mathsf{f}^i}{\partial w_i^3} = 0$ for all $i = 1, \ldots, m$. Therefore, $\mathsf{f}^i(x, w_i) = A_i(x)w_i^2 + B_i(x)w_i + C_i(x)$ and using again the first relation of (5.10) we deduce $A_i\mathtt{I}_i^i - A_j\mathtt{I}_j^j = 0$ for all $1 \le i, j \le m$, implying that $\mathsf{f}(x,w) = A(x)\sum_{i=1}^m w_i^2\mathtt{I}_i^i +$

$B_i(x)w_i + C_i(x)$. Finally, the drift vector field $f = \mathsf{f}\frac{\partial}{\partial x}$ of Σ has the following form (modulo the w-components):

$$f(x,w) = A(x)w^t \mathtt{I}_{p,q} w + B(x)w + C(x),$$

where $B = (B_1, \ldots, B_m)$, and $C = \sum_{i=1}^m C_i$. For Σ in this form we have $\mathcal{D}^1 = \mathcal{D}^0 + \operatorname{span}\left\{(2Aw_i\mathtt{I}_i^i + B_i)\frac{\partial}{\partial x},\ i = 1, \ldots, m\right\}$ which, by assumption (A2), is of constant rank $2m$, and since we have preserved the signature of Ω_ω in all our operations, we obtain that $A\frac{\partial}{\partial x} \notin \mathcal{D}^1$ and thus we conclude $A \wedge B_1 \wedge \ldots \wedge B_m \neq 0$. Hence, we have completed transforming of Σ into a (p,q)-paraboloid system $\Sigma_{p,q}$. ■

Our characterisation of (p,q)-paraboloid systems requires two kinds of conditions. The first kind, given by the two conditions (5.7a) and (5.7b), is purely algebraic and is related to the (p,q)-paraboloid structure of $\Sigma_{p,q}$. The second kind, given by conditions (5.9a) and (5.9b), is differential and asserts that there exists a feedback $\beta(\cdot) \in GO(p,q)$ that transforms a WQF into a SQF. Our definition of a SQF (which is the key of our characterisation of(p,q)-systems) is of second order with respect to w but, as we observe in the proof, it implies a third order condition on the drift vector field of Σ (see below equation (5.10)).

Remark (Comparison with the cases $m = 1$ and $m = 2$). The proof of the implication (S4)⇒(S1) uses the alternative definition (5.5) of a strong quadratic frame. This definition requires at least two controls (i.e. two vector fields g_i) to be nontrivial. We therefore understand why the case $m = 1$ has to be treated separately and why for that case we need to compute third order Lie brackets (see Theorem 2.2 of Chapter 2), while for the cases $m \geq 2$ we only need second order brackets.
Compare the above theorem with Theorems 4.4 and 4.10 of Chapter 4, which are clearly special cases of the general result formulated in Theorem 5.3 (p-elliptic system being $\Sigma_{2,0}$ and p-hyperbolic systems being $\Sigma_{1,1}$). The only small difference is in the definitions of WQF and SQF, called weak/strong orthonormal or isotropic frames in Chapter 4, where we directly used our alternative definition (given by relation (5.5)) and we did not introduce the definition with the vector field Z. The approach of Chapter 4 leads to a different definition of the structure functions μ, but all in all the conditions that we obtained are of similar nature. ◆

In this section, we introduced the notion of weak and strong quadratic frame attached to a control-affine system Σ, satisfying assumptions (A1), (A2), and (A3). We showed that the existence of a strong quadratic frame can explicitly be checked by means of algebraic and differential relations between structure functions. Our main result shows that the existence of a strong quadratic frame is a complete characterisation of (p,q)-paraboloid systems $\Sigma_{p,q}$. In the following section, we will analyse the class of (p,q)-paraboloid systems and study the classification problem.

2 Classification of (p,q)-paraboloid systems

We now investigate the problem of classifying the class of (p,q)-paraboloid submanifolds $\mathcal{S}_Q$ of $T\mathcal{X}$. This problem is dealt with under the classification of their first prolongations (regular parametrisations) defined by

$$\Xi_{p,q} \,:\, \dot{x} = A(x)w^t \mathtt{I}_{p,q} w + \sum_{i=1}^{m} B_i(x)w_i + C(x),$$

where the state x belongs to $\mathcal{X}$, a smooth manifold of dimension $m+1$, $w = (w_1, \dots, w_m)^t \in \mathbb{R}^m$ is the control that enters in a quadratic way, and $\mathtt{I}_{p,q} = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}$ (without loss of generality, we assume that $p \geq q$). Moreover, A, $B = (B_1, \dots, B_m)$, and C are smooth vector fields satisfying $A \wedge B_1 \wedge \dots \wedge B_m \neq 0$ in a neighbourhood of x_0. A (p,q)-paraboloid control-nonlinear system $\Xi_{p,q}$, shortly a (p,q)-system, is represented by the $(m{+}2)$-tuple of vector fields $(A, B_1, \dots, B_m, C) = (A, B, C)$. We will describe several orbits of $\Xi_{p,q}$ under the action of feedback transformations $\tilde{x} = \phi(x)$ and $w = \psi(x, \tilde{w})$. Clearly, if two systems $\Xi_{p,q}$ and $\tilde{\Xi}_{\tilde{p},\tilde{q}}$ are feedback equivalent, then $(p,q) = (\tilde{p}, \tilde{q})$ (under our convention that $p \geq q$ and $\tilde{p} \geq \tilde{q}$), so we assume that (p,q) is fixed. First of all, we have the following characterisation of feedback transformations that transfer a (p,q)-system into another.

Proposition 5.3 (Equivalence of (p,q)-paraboloid control-nonlinear systems)**.**

(i) *If two (p,q)-systems $\Xi_{p,q} = (A, B, C)$ and $\tilde{\Xi}_{p,q} = (\tilde{A}, \tilde{B}, \tilde{C})$ are feedback equivalent via a diffeomorphism $\tilde{x} = \phi(x)$ and an invertible feedback transformation $w = \psi(x, \tilde{w})$, then $\psi(x, \tilde{w}) = \alpha(x) + \beta(x)\tilde{w}$ where $\alpha \in C^\infty(\mathcal{X}, \mathbb{R}^m)$ and $\beta \in C^\infty(\mathcal{X}, GO(p,q))$, i.e. $\beta^t I_{p,q} \beta = \lambda\, I_{p,q}$ with λ a smooth function satisfying $\lambda(\cdot) \neq 0$. Moreover, we have*

$$\tilde{A} = \phi_*(\lambda A), \quad \tilde{B} = \phi_*(2A\,\alpha^t I_{p,q}\beta + B\beta), \quad \text{and} \quad \tilde{C} = \phi_*(C + A\,\alpha^t I_{p,q}\alpha + B\alpha). \tag{5.11}$$

(ii) *Conversely, if a diffeomorphism $\tilde{x} = \phi(x)$ and smooth functions $\alpha : \mathcal{X} \to \mathbb{R}^m$ and $\beta : \mathcal{X} \to GO(p,q)$ satisfy (5.11), then the feedback transformation $\tilde{x} = \phi(x)$ together with $\psi(x, \tilde{w}) = \alpha(x) + \beta(x)\tilde{w}$ transforms $\Xi_{p,q}$ into $\tilde{\Xi}_{p,q}$.*

The implicit summations in formula (5.11) for $\tilde{B}$ and $\tilde{C}$ should be interpreted as follows (where we take $\phi = \mathrm{Id}$ to simplify the notations):

$$\tilde{B}_k = 2A\sum_{i=1}^m \alpha^i \mathtt{I}_i^i \beta_k^i + \sum_{i=1}^m B_i\beta_k^i \quad \text{and} \quad \tilde{C} = C + A\sum_{i=1}^m \mathtt{I}_i^i(\alpha^i)^2 + \sum_{i=1}^m B_i\alpha^i.$$

Remark (Local character of the results). When we introduced the definition of (p,q)-paraboloid nonlinear systems $\Xi_{p,q}$, we assumed that this form holds locally around an arbitrary point(x_0, w_0). We see, in statement *(i)* of the above proposition that the pure feedback transformation $w = \psi(x, \tilde{w})$ that conjugate (p,q)-paraboloid systems is global with respect to the control w. Therefore, in all results below, we will consider the form $\Xi_{p,q}$ *locally* around x_0 and *globally* in w. ◆

Proof.

(i) Clearly diffeomorphisms of $\mathcal{X}$ map (p,q)-systems into (p,q)-systems and we have to show that only pure feedback transformations $w = \psi(x, \tilde{w})$ of the form $w = \alpha(x) + \beta(x)\tilde{w}$ conjugate (p,q)-systems. To this end, applying $(w_1, \dots, w_m) = (\psi_1(x, \tilde{w}), \dots, \psi_m(x, \tilde{w}))$ to $\Xi_{p,q}$ yields

$$\dot{x} = A\,\psi^t \mathtt{I}_{p,q}\psi + B\psi + C. \tag{5.12}$$

To ensure the quadratic structure of $\tilde{\Xi}_{p,q}$, we must have

$$\frac{\partial^3}{\partial \tilde{w}_i \partial \tilde{w}_j \partial \tilde{w}_k} \left(A\,\psi^t \mathtt{I}_{p,q} \psi + B\psi \right) = 0, \qquad \text{for all} \quad i, j, k.$$

Since the vector fields $A = A(x)$ and $B_i = B_i(x)$, for $i = 1, \ldots, m$, are linearly independent for every x, we first conclude that $\frac{\partial^3\, \psi}{\partial \tilde{w}_i \partial \tilde{w}_j \partial \tilde{w}_k} = 0$, i.e. ψ is a polynomial of degree at most 2 in $\tilde{w}$. Secondly, by equating the indices $i = j = k$, we deduce that $\frac{\partial \psi^t}{\partial w_i} \mathtt{I}_{p,q} \frac{\partial^2 \psi}{\partial w_i^2} = 0$ implying that ψ is, actually, affine with respect to $\tilde{w}$, i.e. $\psi(x, \tilde{w}) = \beta(x)\tilde{w} + \alpha(x)$ with $\beta \in C^\infty(\mathcal{X}, GL(m))$ and $\alpha \in C^\infty(\mathcal{X}, \mathbb{R}^m)$. Finally, with (5.12) we obtain that β satisfies $\beta^t \mathtt{I}_{p,q} \beta = \lambda\, \mathtt{I}_{p,q}$ in order to get $A\tilde{w}^t \beta^t \mathtt{I}_{p,q} \beta \tilde{w} = \tilde{A}\tilde{w}^t \mathtt{I}_{p,q} \tilde{w}$. In order to obtain formula (5.11) we apply (5.12) with $\psi = \beta\tilde{w} + \alpha$ satisfying $\beta^t \mathtt{I}_{p,q} \beta = \lambda\, \mathtt{I}_{p,q}$.

(ii) Conversely, for ϕ and (α, β) satisfying (5.11), we clearly establish feedback equivalence of $\Xi_{p,q}$ and $\tilde{\Xi}_{p,q}$ via $\tilde{x} = \phi(x)$ and $w = \beta\tilde{w} + \alpha$.

■

Remark (Nature of the feedback transformations). Pure feedback transformations acting on (p, q)-systems are of the form $w = \beta\tilde{w} + \alpha$, with $\beta(x) \in GO(p, q)$ and $\alpha(x) \in \mathbb{R}^m$. In particular, β can be a homothety $\beta = \eta\, \mathrm{Id}_m$, for some smooth function $\eta \neq 0$, or an indefinite isometry $\beta \in O(p, q)$, but in general β is not a composition of those two operations. Homotheties act on $\Xi_{p,q} = (A, B, C)$ by scaling: $(\tilde{A}, \tilde{B}, \tilde{C}) = (\eta^2 A, \eta B, C)$, and indefinite isometries act by indefinite rotations on the fields B: $(\tilde{A}, \tilde{B}, \tilde{C}) = (A, B\beta, C)$. The action of α leaves the field A invariant and changes the directions of B and C by $(\tilde{A}, \tilde{B}, \tilde{C}) = (A, B + 2A\alpha^t \mathtt{I}_{p,q}, C + A\alpha^t \mathtt{I}_{p,q}\alpha + B\alpha)$. ◆

Feedback transformations $\psi(x, \tilde{w}) = \beta(x)\tilde{w} + \alpha(x)$ are denoted by the couple (α, β). We will develop relations involving structure functions uniquely attached to the $(m+2)$-tuple (A, B, C) only, and thus independent from diffeomorphisms of $\mathcal{X}$. So we will act on $\Xi_{p,q} = (A, B, C)$ by (α, β) and we will denote $(\tilde{A}, \tilde{B}, \tilde{C})$ the result of that action (given by (5.11) with $\phi = \mathrm{Id}$) and call it a *reparametrisation*. For a (p, q)-system $\Xi_{p,q}$ we call the $(m+1)$-tuple (A, B) a (p, q)-frame and we introduce the structure functions μ_j^0, μ_j^i, $\nu_{k,j}^0$, and $\nu_{k,j}^i$ for $i, j, k = 1, \ldots, m$ defined by the following brackets

$$[A, B_j] = A\mu_j^0 + \sum_{i=1}^m B_i \mu_j^i, \quad \text{and} \quad [B_k, B_j] = A\nu_{k,j}^0 + \sum_{i=1}^m B_i \nu_{k,j}^i.$$

Notice that symbols similar to those of the previous section are used although they have nothing to do with each other. Moreover, we define the structure functions γ^0 and γ^i, for $i = 1, \ldots, m$, given by the decomposition of the field C in the frame (A, B):

$$C = \gamma^0 A + \sum_{i=1}^m B_i \gamma^i.$$

By a simple calculation, we see that we defined $\frac{m(m+1)^2}{2} + (m+1)$ unique structure functions (which are related by the Jacobi identity). To shorten our formulae, we exploit the matrix-vector product with the following notations $\mu^0 := (\mu^0_1, \ldots, \mu^0_m)$, $\mu := (\mu^i_j)$, $\nu^0_k := (\nu^0_{k,1}, \ldots, \nu^0_{k,m})$, $\nu_k := (\nu^i_{k,j})$, and $\gamma = (\gamma^1, \ldots, \gamma^m)^t$. Thus, the above definitions can be rewritten as

$$[A, B] = A\mu^0 + B\mu, \quad [B_k, B] = A\nu^0_k + B\nu_k, \quad \text{and} \quad C = A\gamma^0 + B\gamma.$$

We now introduce different types of (p, q)-frames. We denote the distribution $\mathcal{A} :=$ span $\{A\}$, which from the first relation of (5.11) is seen to be invariant under feedback transformations, i.e. the distribution $\mathcal{A}$ is uniquely attached to a (p, q)-system.

Definition 5.4 (Types of (p, q)-frames)**.** We say that a (p, q)-frame (A, B), with structure functions $\{\mu^0, \mu, \nu^0_k, \nu_k\}$, is

(a) *pseudo-commutative* if $[A, B] = 0 \mod \mathcal{A}$, that is $\mu = 0$.

(b) *almost-commutative* if $[A, B] = [B_k, B] = 0 \mod \mathcal{A}$, that is $\mu = 0$ and, additionally, $\nu_k = 0$ for $k = 1, \ldots, m$.

(c) *commutative* if $[A, B] = [B_k, B] = 0$, that is $\mu = \nu_k = 0$ and, additionally, $\mu^0 = \nu^0_k = 0$ for $k = 1, \ldots, m$.

Clearly, pseudo-commutative frames form a subclass of almost-commutative frames, which themselves form a subclass of commutative frames. Actually, we will prove in Theorem 5.5 that pseudo-commutative and almost-commutative frames form the same class. The following technical lemma shows how the six sets of structure functions μ^0_j, μ^i_j, $\nu^0_{k,j}$, $\nu^i_{k,j}$, γ^0, and γ^i are transformed under reparametrisations (α, β). Relation (5.14) for $\tilde{\nu}_k$ is given for completeness only and will not be used in what follows.

Lemma 5.1 (Structure functions transformations)**.** *Let $\Xi_{p,q} = (A, B, C)$ and $\tilde{\Xi}_{p,q} = (\tilde{A}, \tilde{B}, \tilde{C})$ be two (p, q)-paraboloid nonlinear systems with structure functions μ^0_j, μ^i_j, $\nu^0_{k,j}$, $\nu^i_{k,j}$, γ^0, γ^i and $\tilde{\mu}^0_j$, $\tilde{\mu}^i_j$, $\tilde{\nu}^0_{k,j}$, $\tilde{\nu}^i_{k,j}$, $\tilde{\gamma}^0$, $\tilde{\gamma}^i$, respectively. Suppose that they are feedback equivalent via (α, β), with $\beta^t I_{p,q} \beta = \lambda I_{p,q}$, then we have*

$$\begin{aligned} \lambda\tilde{\mu}^0 + 2\alpha^t I_{p,q}\beta\tilde{\mu} &= \lambda\mu^0\beta - \mathrm{L}_B(\lambda)\,\beta + 2\lambda\mathrm{L}_A\left(\alpha^t I_{p,q}\beta\right) - 2\alpha^t I_{p,q}\beta\mathrm{L}_A(\lambda)\,, \\ \beta\tilde{\mu} &= \lambda\mu\beta + \lambda\mathrm{L}_A(\beta)\,, \end{aligned} \tag{5.13}$$

$$\begin{aligned} \lambda\tilde{\nu}^0_{k,j} + 2\alpha^t I_{p,q}\beta\tilde{\nu}_{k,j} &= 2\alpha^t I_{p,q}\beta_k\left[\mu^0\beta + 2\mathrm{L}_A\left(\alpha^t I_{p,q}\beta\right)\right]_j \\ &\quad -2\alpha^t I_{p,q}\beta_j\left[\mu^0\beta + 2\mathrm{L}_A\left(\alpha^t I_{p,q}\beta\right)\right]_k \\ &\quad +2\mathrm{L}_B\left(\alpha^t I_{p,q}\beta_j\right)\beta_k \\ &\quad -2\mathrm{L}_B\left(\alpha^t I_{p,q}\beta_k\right)\beta_j \\ &\quad +\textstyle\sum_{s,r=1}^m \beta^s_k\nu^0_{s,r}\beta^r_j\,, \\ \left(\beta\tilde{\nu}_k\right)^i_j &= 2\alpha^t I_{p,q}\left(\beta_k\left[\mu\beta + \mathrm{L}_A(\beta)\right]^i_j - \left[\mu\beta + \mathrm{L}_A(\beta)\right]^i_k\beta_j\right) \\ &\quad +\mathrm{L}_B\left(\beta^i_j\right)\beta_k - \mathrm{L}_B\left(\beta^i_k\right)\beta_j + \textstyle\sum_{s,r=1}^m \beta^s_k\nu^i_{s,r}\beta^r_j\,, \end{aligned} \tag{5.14}$$

$$\tilde{\gamma}^0 = \tfrac{1}{\lambda}\left(\gamma^0 - \alpha^t I_{p,q}\alpha - 2\alpha^t I_{p,q}\gamma\right) \qquad \text{and} \qquad \tilde{\gamma} = \beta^{-1}(\gamma + \alpha)\,. \tag{5.15}$$

Moreover, the following relations between the structure functions always hold (we use implicit summation over the index l). For all $1 \leq k < j \leq m$ and all $1 \leq i \leq m$:

$$\begin{aligned} \mathrm{L}_A\left(\nu^0_{k,j}\right) - \mathrm{L}_{B_k}\left(\mu^0_j\right) + \mathrm{L}_{B_j}\left(\mu^0_k\right) &= -\mu^0_l\nu^l_{k,j} + \nu^0_{k,l}\mu^l_j + \mu^l_k\nu^0_{l,j}, \\ \mathrm{L}_A\left(\nu^i_{k,j}\right) - \mathrm{L}_{B_k}\left(\mu^i_j\right) + \mathrm{L}_{B_j}\left(\mu^i_k\right) &= -\mu^i_l\nu^l_{k,j} + \nu^i_{k,l}\mu^l_j + \mu^l_k\nu^i_{l,j} + \mu^i_k\mu^0_j + \mu^i_j\mu^0_k, \end{aligned} \tag{5.16a}$$

furthermore, for all $1 \le i < k < j \le m$ *and all* $1 \le s \le m$,

$$\text{(5.16b)}\quad \begin{aligned} \mathrm{L}_{B_i}\left(\nu^0_{k,j}\right) + \mathrm{L}_{B_k}\left(\nu^0_{j,i}\right) + \mathrm{L}_{B_j}\left(\nu^0_{i,k}\right) &= \mu^0_i \nu^0_{k,j} + \mu^0_k \nu^0_{j,i} + \mu^0_j \nu^0_{i,k} \\ &\quad - \nu^0_{i,l}\nu^l_{k,j} - \nu^0_{k,l}\nu^l_{j,i} - \nu^0_{j,l}\nu^l_{i,k}, \\ \mathrm{L}_{B_i}\left(\nu^s_{k,j}\right) + \mathrm{L}_{B_k}\left(\nu^s_{j,i}\right) + \mathrm{L}_{B_j}\left(\nu^s_{i,k}\right) &= \mu^s_i \nu^0_{k,j} + \mu^s_k \nu^0_{j,i} + \mu^s_j \nu^0_{i,k} \\ &\quad - \nu^s_{i,l}\nu^l_{k,j} - \nu^s_{k,l}\nu^l_{j,i} - \nu^s_{j,l}\nu^l_{i,k}. \end{aligned}$$

Formulae (5.16a) and (5.16b) can be rewritten using our matrix notations as follows. For all indices $1 \le k \le m$ it holds

$$\text{(5.16a')}\quad \begin{aligned} \mathrm{L}_A\left(\nu^0_k\right) - \mathrm{L}_{B_k}\left(\mu^0\right) + \mathrm{L}_B\left(\mu^0_k\right) &= -\mu^0\nu_k + \nu^0_k \mu + \textstyle\sum_{l=1}^m \mu^l_k \nu^0_l, \\ \mathrm{L}_A\left(\nu_k\right) - \mathrm{L}_{B_k}\left(\mu\right) + \mathrm{L}_B\left(\mu_k\right) &= \nu_k\mu - \mu\nu_k + \mu_k\mu^0 + \mu\mu^0_k + \textstyle\sum_{l=1}^m \mu^l_k \nu_l, \end{aligned}$$

and, for all $1 \le k < j \le m$ it holds

$$\text{(5.16b')}\quad \begin{aligned} \mathrm{L}_B\left(\nu^0_{kj}\right) + \mathrm{L}_{B_k}\left(\nu^0_j\right) - \mathrm{L}_{B_j}\left(\nu^0_k\right) &= \mu^0\nu^0_{kj} + \mu^0_k\nu^0_j - \mu^0_j\nu^0_k + \nu^0_j\nu_k - \nu^0_k\nu_j \\ &\quad + \textstyle\sum_{l=1}^m \nu^0_l \nu^l_{kj}, \\ \mathrm{L}_B\left(\nu_{kj}\right) + \mathrm{L}_{B_k}\left(\nu_j\right) - \mathrm{L}_{B_j}\left(\nu_k\right) &= \mu\nu^0_{kj} + \mu_k\nu^0_j - \mu_j\nu^0_k + \nu_k\nu_j - \nu_j\nu_k \\ &\quad + \textstyle\sum_{l=1}^m \nu_l \nu^l_{kj}. \end{aligned}$$

Formulae (5.13), (5.14), (5.15), and (5.16a) are generalisation of those obtained in the case $m = 2$ (compare Lemmas 4.1 and 4.2 of Chapter 4). However, the relations of (5.16b) are new because they come from the Jacobi identity between three different vector fields B_i.

Proof. Let $\Xi_{p,q} = (A, B, C)$ and $\tilde{\Xi}_{p,q} = (\tilde{A}, \tilde{B}, \tilde{C})$ be two (p,q)-systems with structure functions μ^0_j, μ^i_j, $\nu^0_{k,j}$, $\nu^i_{k,j}$, γ^0, γ^i and $\tilde{\mu}^0_j$, $\tilde{\mu}^i_j$, $\tilde{\nu}^0_{k,j}$, $\tilde{\nu}^i_{k,j}$, $\tilde{\gamma}^0$, $\tilde{\gamma}^i$, respectively, feedback equivalent via (α, β). Recall from (5.11), the following relations that we will systematically use: $\tilde{A} = \lambda A$ and $\tilde{B} = 2A\,\alpha^t \mathtt{I}_{p,q}\beta + B\beta$.
We begin by showing how the structure functions μ^0 and μ are transformed:

$$\begin{aligned} \tilde{A}\tilde{\mu}^0 + \tilde{B}\tilde{\mu} &= \left[\tilde{A}, \tilde{B}\right] = \left[\lambda A, B\beta + 2A\,\alpha^t \mathtt{I}_{p,q}\beta\right], \\ A\left(\lambda\tilde{\mu}^0 + 2\alpha^t \mathtt{I}_{p,q}\beta\tilde{\mu}\right) + B\beta\tilde{\mu} &= \lambda\left[A, B\right]\beta + \lambda B \mathrm{L}_A\left(\beta\right) - A\mathrm{L}_B\left(\lambda\right)\beta \\ &\quad + \lambda A \mathrm{L}_A\left(2\alpha^t \mathtt{I}_{p,q}\beta\right) - 2A\alpha^t \mathtt{I}_{p,q}\beta \mathrm{L}_A\left(\lambda\right), \\ &= A\left(\lambda\mu^0\beta - \mathrm{L}_B\left(\lambda\right)\beta + \lambda\mathrm{L}_A\left(2\alpha^t \mathtt{I}_{p,q}\beta\right) - 2\alpha^t \mathtt{I}_{p,q}\beta\mathrm{L}_A\left(\lambda\right)\right) \\ &\quad + B\left(\lambda\mu\beta + \lambda\mathrm{L}_A\left(\beta\right)\right). \end{aligned}$$

Identifying the terms in front of A and B implies formula (5.13). We continue our calculations by showing how the structure functions $\nu^0_{k,j}$ and $\nu^i_{k,j}$ are transformed:

$$\begin{aligned} \left[\tilde{B}_k, \tilde{B}_j\right] &= \tilde{A}\tilde{\nu}^0_{k,j} + \tilde{B}\tilde{\nu}_{k,j} = A\left(\lambda\tilde{\nu}^0_{k,j} + 2\alpha^t \mathtt{I}_{p,q}\beta\tilde{\nu}_{k,j}\right) + \sum_{i=1}^m B_i\beta^i\tilde{\nu}_{k,j}, \\ &= \left[2A\alpha^t \mathtt{I}_{p,q}\beta_k + B\beta_k, 2A\alpha^t \mathtt{I}_{p,q}\beta_j + B\beta_j\right], \\ &= 2\alpha^t \mathtt{I}_{p,q}\beta_k\left[A, B\right]\beta_j + 2\alpha^t \mathtt{I}_{p,q}\beta_j\left[B, A\right]\beta_k + \sum_{s,r=1}^m \left[B_s\beta^s_k, B_r\beta^r_j\right] \\ &\quad + 4\alpha^t \mathtt{I}_{p,q}\beta_k A\mathrm{L}_A\left(\alpha^t \mathtt{I}_{p,q}\beta_j\right) - 4\alpha^t \mathtt{I}_{p,q}\beta_j A\mathrm{L}_A\left(\alpha^t \mathtt{I}_{p,q}\beta_k\right) \\ &\quad + 2A\mathrm{L}_B\left(\alpha^t \mathtt{I}_{p,q}\beta_j\right)\beta_k - 2A\mathrm{L}_B\left(\alpha^t \mathtt{I}_{p,q}\beta_k\right)\beta_j \\ &\quad + 2\alpha^t \mathtt{I}_{p,q}\beta_k B\mathrm{L}_A\left(\beta_j\right) - 2\alpha^t \mathtt{I}_{p,q}\beta_j B\mathrm{L}_A\left(\beta_k\right), \end{aligned}$$

implying

$$
\begin{aligned}
&= A\Bigg(2\alpha^t\mathtt{I}_{p,q}\beta_k\mu^0\beta_j - 2\alpha^t\mathtt{I}_{p,q}\beta_j\mu^0\beta_k + \sum_{s,r=1}^{m}\beta_k^s\beta_j^r\nu_{s,r}^0 \\
&+ 4\alpha^t\mathtt{I}_{p,q}\beta_k\mathrm{L}_A\left(\alpha^t\mathtt{I}_{p,q}\beta_j\right) - 4\alpha^t\mathtt{I}_{p,q}\beta_j\mathrm{L}_A\left(\alpha^t\mathtt{I}_{p,q}\beta_k\right) \\
&+ 2\mathrm{L}_B\left(\alpha^t\mathtt{I}_{p,q}\beta_j\right)\beta_k - 2\mathrm{L}_B\left(\alpha^t\mathtt{I}_{p,q}\beta_k\right)\beta_j\Bigg) \\
&+ \sum_{i=1}^{m} B_i\Bigg(2\alpha^t\mathtt{I}_{p,q}\beta_k\mu^i\beta_j - 2\alpha^t\mathtt{I}_{p,q}\beta_j\mu^i\beta_k + \sum_{s,r=1}^{m}\beta_k^s\beta_j^r\nu_{s,r}^i \\
&+ 2\alpha^t\mathtt{I}_{p,q}\beta_k\mathrm{L}_A\left(\beta_j^i\right) - 2\alpha^t\mathtt{I}_{p,q}\beta_j\mathrm{L}_A\left(\beta_k^i\right) \\
&+ \sum_{s=1}^{m}\beta_k^s\mathrm{L}_{B_s}\left(\beta_j^i\right) - \sum_{r=1}^{m}\beta_j^r\mathrm{L}_{B_r}\left(\beta_k^i\right)\Bigg),
\end{aligned}
$$

implying formula (5.14). We conclude our calculations with the transformation of the structure functions γ^0 and γ^i:

$$
\begin{aligned}
\tilde{A}\tilde{\gamma}^0 + \tilde{B}\tilde{\gamma} = \tilde{C} = C + A\,\alpha^t\mathtt{I}_{p,q}\alpha + B\alpha &= A\gamma^0 + B\gamma + A\,\alpha^t\mathtt{I}_{p,q}\alpha + B\alpha, \\
A\left(\lambda\tilde{\gamma}^0 + 2\alpha^t\mathtt{I}_{p,q}\beta\tilde{\gamma}\right) + B\beta\tilde{\gamma} &= A\left(\gamma^0 + \alpha^t\mathtt{I}_{p,q}\alpha\right) + B\left(\alpha+\gamma\right),
\end{aligned}
$$

from which we deduce first $\tilde{\gamma} = \beta^{-1}(\gamma+\alpha)$, and then $\tilde{\gamma}^0 = \frac{1}{\lambda}\left(\gamma^0 - \alpha^t\mathtt{I}_{p,q}\alpha - 2\gamma^t\mathtt{I}_{p,q}\alpha\right)$. To show (5.16a), we apply the Jacobi identity to $[A,[B_k,B]]$ and identify the terms in front of A and B, as follows:

$$
\begin{aligned}
[A,[B_k,B]] - [B_k,[A,B]] + [B,[A,B_k]] &= 0, \\
\left[A, A\nu_k^0 + B\nu_k\right] - \left[B_k, A\mu^0 + B\mu\right] + \left[B, A\mu_k^0 + B\mu_k\right] &= 0, \\
\mathrm{L}_A\left(\nu_k^0\right)A + [A,B]\,\nu_k + B\mathrm{L}_A\left(\nu_k\right) + [A,B_k]\,\mu^0 - A\mathrm{L}_{B_k}\left(\mu^0\right) - [B_k,B]\,\mu & \\
-B\mathrm{L}_{B_k}\left(\mu\right) - [A,B]\,\mu_k^0 + A\mathrm{L}_B\left(\mu_k^0\right) + \sum_{l=1}^{m}\left[B, B_l\mu_k^l\right] &= 0.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
A(\mathrm{L}_A\left(\nu_k^0\right) + \mu^0\nu_k + \cancel{\mu_k^0\mu^0} - \mathrm{L}_{B_k}\left(\mu^0\right) - \nu_k^0\mu - \cancel{\mu^0\mu_k^0} + \mathrm{L}_B\left(\mu_k^0\right) - \sum_{l=1}^{m}\nu_l^0\mu_k^l) \\
+B(\mu\nu_k + \mathrm{L}_A\left(\nu_k\right) + \mu_k\mu^0 - \nu_k\mu - \mathrm{L}_{B_k}\left(\mu\right) - \mu\mu_k^0 + \mathrm{L}_B\left(\mu_k\right) - \sum_{l=1}^{m}\nu_l\mu_k^l) = 0,
\end{aligned}
$$

implying the two equations of (5.16a') which can be expended in the formulae of (5.16a). Finally, to show (5.16b) we use the Jacobi identity on $[B_i,[B_k,B]]$ and identify the term in front of A and B_s (for all $1 \leq s \leq m$). ■

We now begin to give our classification of control-nonlinear (p,q)-systems. Our conditions will be expressed by relations between the structure functions (except for the next result, where only geometric conditions are given). The following proposition gives a geometric characterisation of the following normal form

$$\Xi^d_{p,q} \,:\, \dot{x} = w^t \mathrm{I}_{p,q} w \frac{\partial}{\partial z} + \sum_{i=1}^{m} q_i(x) w_i \frac{\partial}{\partial y_i} + C(x),$$

which is equivalent (via the transformation $\tilde{w}_i = q_i(x)w_i + c_i(x)$) to a first prolongation of (p,q)-paraboloid submanifold $\mathcal{S}_Q$ with the matrix $Q(x)$ being locally diagonalised.

Proposition 5.4. *Consider a (p,q)-system $\Xi_{p,q} = (A, B, C)$, then the following statements are equivalent.*

(i) $\Xi_{p,q}$ is feedback equivalent to $\Xi^d_{p,q}$,

(ii) There exists a reparametrisation (α, β) of (A, B) such that

$$\left[\tilde{A}, \tilde{B}_j\right] = 0 \bmod \operatorname{span}\left\{\tilde{A}, \tilde{B}_j\right\} \quad \text{and} \quad \left[\tilde{B}_k, \tilde{B}_j\right] = 0 \bmod \operatorname{span}\left\{\tilde{A}, \tilde{B}_k, \tilde{B}_j\right\}. \tag{5.17}$$

Compare the above proposition with Proposition 4.4 of Chapter 4 and observe that when $m = 2$, then the second condition of (5.17) is automatically fulfilled. Contrary to the case of $m = 2$, we did not succeed in finding a condition on the structure functions ensuring the existence of a feedback transformation (α, β) achieving (5.17).

Remark. The geometric meaning of condition (5.17) is as follows. We define the projection π onto the space of leaves of the foliation generated by $\mathcal{A}$, by attaching to $x \in \mathcal{X}$ the leaf of the integral foliation of $\mathcal{A}$ passing through x, and its tangent map π_* acting on the vector fields B_i:

$$\begin{aligned} \pi_* \,:\, T\mathcal{X} &\longrightarrow T\mathcal{X}/\mathcal{A} \\ B_i &\longmapsto \pi_* B_i. \end{aligned}$$

The first equality of (5.17) implies (see the proof below) that the distributions $\hat{\mathcal{B}}_i = \pi_*\mathcal{B}_i$, where $\mathcal{B}_i = \operatorname{span}\{B_i\}$, are well defined (although the vector fields $\pi_* B_i$ need not be). The second equation of (5.17) is the key of our characterisation, it implies that the distributions $\hat{\mathcal{B}}_k \oplus \hat{\mathcal{B}}_j$ are involutive. Using a generalisation of Frobenius theorem (that we propose in Appendix A), we conclude that the distributions $\hat{\mathcal{B}}_i$, for $1 \leq i \leq m$, are simultaneously rectifiable. ◆

Proof. The proof *(i)*⇒*(ii)* is immediate by a straightforward calculation.
(ii)⇒*(i)*. Assume that the (p,q)-frame (A, B) of $\Xi_{p,q}$ satisfies (5.17). Introduce local coordinates $\tilde{x} = (z, \tilde{y}) = \phi(x)$, around $x_0 \in \mathcal{X}$, such that $\phi_* A = \frac{\partial}{\partial z}$, and set $\tilde{B}_i = \phi_* B_i = b^0_i(\tilde{x})\frac{\partial}{\partial z} + \sum_{j=1}^m \tilde{b}^j_i(\tilde{x})\frac{\partial}{\partial \tilde{y}_j}$. By assumption, there exists smooth functions $(\rho_1, \ldots, \rho_m)$ such that $\rho_i\left[A, \tilde{B}_i\right] = \tilde{B}_i \mod \mathcal{A}$, implying $\rho_i \frac{\partial \tilde{b}^j_i}{\partial z} = \tilde{b}^j_i$ for all $j = 1, \ldots, m$. By solving those equations, we deduce that $\tilde{b}^j_i(\tilde{x}) = \hat{b}^j_i(\tilde{y})\exp(R_i(\tilde{x}))$. It follows that the family of involutive distributions

$$\hat{\mathcal{B}}_i = \pi_* \operatorname{span}\left\{\tilde{B}_i\right\} = \operatorname{span}\left\{\hat{b}^1_i(\tilde{y})\frac{\partial}{\partial \tilde{y}_1} + \ldots + \hat{b}^m_i(\tilde{y})\frac{\partial}{\partial \tilde{y}_m}\right\} \quad \text{for } 1 \leq i \leq m,$$

is well defined on the manifold $\mathcal{Y} = \mathcal{X}/\sim$ equipped with coordinates $(\tilde{y}_1, \ldots, \tilde{y}_m)$. Here «$\sim$» is defined locally around x_0 by the involutive distribution $\mathcal{A} = \text{span}\left\{\frac{\partial}{\partial z}\right\}$ and $\pi : \mathcal{X} \to \mathcal{Y}$ is the projection attaching to $x \in \mathcal{X}$ the leaf passing through x of the foliation generate by $\mathcal{A}$. The second relation of (5.17) implies that the distributions $\hat{\mathcal{B}}_i$ satisfy the assumptions of our generalisation of Frobenius theorem (given by Theorem A.2 of Appendix A), namely $\hat{\mathcal{B}}_i \oplus \hat{\mathcal{B}}_j$ is involutive for any $1 \leq i, j \leq m$. Therefore, those distributions can be simultaneously rectified, that is there exists coordinates $y = \varphi(\tilde{y})$ such that $\hat{\mathcal{B}}_i = \text{span}\left\{\frac{\partial}{\partial y_i}\right\}$. In the coordinate system $x = (z, y)$, the system $\Xi_{p,q}$ takes the form

$$\begin{cases} \dot{z} &= w^t \mathtt{I}_{p,q} w + \sum_{i=1}^m b_i^0 w_i + c_0 \\ \dot{y}_i &= q_i w_i + c_i \end{cases},$$

where $q_i = q_i(z, y)$. The conclusion follows from the application of the feedback $w_i = \tilde{w}_i - \frac{1}{2} b_i^0 \mathtt{I}_i^i$. ■

In the following result, we will characterise the normal form

$$\Xi'_{p,q} : \dot{x} = w^t \mathtt{I}_{p,q} w \frac{\partial}{\partial z} + \sum_{i=1}^m w_i \frac{\partial}{\partial y_i} + C(x).$$

For (p, q)-systems $\Xi_{p,q}$, that form corresponds to the existence of a commutative (p, q)-frame (A, B) and, from the point of view of paraboloid submanifolds $\mathcal{S}_Q$, that form corresponds to the normalisation of Q to the constant matrix $\mathtt{I}_{p,q}$.

To characterise $\Xi'_{p,q}$, we start by observing that for any pseudo-commutative (p, q)-frame (A, B), recall $[A, B] = 0 \mod \mathcal{A}$, the projection π_* (defined above) yields well defined vector fields $\pi_* B_i$, for $1 \leq i \leq m$, on the manifold $\mathcal{Y} = \pi(\mathcal{X})$ (locally defined around a fixed $x_0 \in \mathcal{X}$). In that case, we define on $\mathcal{Y}$ a pseudo-Riemannian metric $\mathtt{g}_B$ by declaring the frame $(\pi_* B_1, \ldots, \pi_* B_m)$ pseudo-orthonormal, i.e. for all $i, j = 1, \ldots, m$ we set $\mathtt{g}_B(\pi_* B_i, \pi_* B_j) = \mathtt{I}_j^i$, implying that $\mathtt{g}_B = \mathtt{I}_{p,q}$ in that frame. Notice that any reparametrisation (α, β) that preserves the pseudo-commutativity of the (p, q)-frame satisfies $\mathrm{L}_A(\beta) = 0$ (as it can be deduced from relation (5.13) with $\tilde{\mu} = \mu = 0$). Under such reparametrisation, for the new pseudo-commutative (p, q)-frame $(\tilde{A}, \tilde{B})$ we have

$$\mathtt{g}_B(\pi_* \tilde{B}_i, \pi_* \tilde{B}_j) = (\beta^t \mathtt{I}_{p,q} \beta)_{i,j} \quad \Longleftrightarrow \quad \mathtt{g}_{\tilde{B}} = \beta^t \mathtt{g}_B \beta = \lambda \, \mathtt{g}_B, \tag{5.18}$$

where the function λ satisfy $\mathrm{L}_A(\lambda) = 0$. Thus observe that two feedback equivalent (p, q)-systems $\Xi_{p,q}$ and $\tilde{\Xi}_{p,q}$ (for which pseudo-commutative (p, q)-frames exist) have respective pseudo-Riemannian metrics $\mathtt{g}_B$ and $\mathtt{g}_{\tilde{B}}$ conformally equivalent. A well-known object in conformal geometry is the Cotton tensor, when $m = 3$, or the Weyl tensor, when $m \geq 4$, which are invariant under conformal transformations (see Chapter 1 for a definition of those tensors). In order to avoid unnecessary distinction between those two cases we call those tensors by the name *conformal* tensor. The conformal tensor can be constructed for any pseudo-Riemannian metric $\mathtt{g}_B$ and its vanishing is equivalent to the fact that $\mathtt{g}_B$ is conformally flat; see [Cot99; Wey18].

Theorem 5.5 (Existence of a commutative (p, q)-frame)**.** *Consider a (p, q)-paraboloid nonlinear system $\Xi_{p,q} = (A, B, C)$ with its (p, q)-frame (A, B). Then, the following statements are locally equivalent,* *(i)* $\Xi_{p,q}$ *is feedback equivalent to* $\Xi'_{p,q}$,

(ii) *There exists a reparametrisation* (α, β) *of* (A, B) *such that* $(\tilde{A}, \tilde{B})$ *is a commutative* (p,q)*-frame.*

(iii) *There exists a reparametrisation* (α, β) *of* (A, B) *such that* $(\tilde{A}, \tilde{B})$ *is an almost-commutative* (p,q)*-frame.*

(iv) *There exists a reparametrisation* (α, β) *of* (A, B) *such that* $(\tilde{A}, \tilde{B})$ *is a pseudo-commutative* (p,q)*-frame and the pseudo-Riemannian metric* $\mathtt{g}_{\tilde{B}}$ *is conformally flat.*

(v) *The structure functions* μ^i_j *of the* (p,q)*-frame* (A, B) *satisfy*

$$\mu^i_i = \mu^j_j, \quad \textit{and} \quad \mu^{\triangle} := \mu - \mathrm{diag}\,(\mu) \in \mathit{Lie}(O(p,q)), \tag{5.19}$$

and the conformal tensor, associated to the pseudo-Riemannian metric $\mathtt{g}_{\tilde{B}}$ *of any equivalent pseudo-commutative* (p,q)*-frame, vanishes.*

The above theorem shows that the class of commutative and of almost-commutative (p,q)-frames are the same, and that the gap between almost- and pseudo-commutative (p,q)-frames is the metric $\mathtt{g}_B$ being conformally flat.

Remark. In statement *(v)*, the relations given by (5.19) form algebraic necessary and sufficient conditions for the existence of a pseudo-commutative (p,q)-frame. Hence, it is necessary to test (5.19) first and next one need to construct a pseudo-commutative (p,q)-frame $(\tilde{A}, \tilde{B})$, same procedure as in the proof below, on which the conformal flatness of $\mathtt{g}_{\tilde{B}}$ can be tested. The condition on the conformal flatness of $\mathtt{g}_{\tilde{B}}$ cannot be tested on an arbitrary (non necessarily pseudo-commutative) (p,q)-frame because in order to define $\mathtt{g}_{\tilde{B}}$ we need the projections $\pi_* B_i$ to be well defined. The construction of a pseudo-commutative (p,q)-frame require solving a differential equation, thus its construction can not always be explicitly done. In the future, we plan to look for a condition expressed on any (p,q)-frame (A, B) which would guarantee that any equivalent pseudo-commutative (p,q)-frame has a conformally flat pseudo-Riemannian metric. ◆

Proof. We will show *(i)*⇒*(v)*⇒*(iv)*⇒*(iii)*⇒*(ii)*⇒*(i)*.
(i)⇒*(v)*. Assume that $\Xi_{p,q}$, given with its (p,q)-frame (A, B), is feedback equivalent to $\Xi'_{p,q}$, given with its (p,q)-frame $(\tilde{A}, \tilde{B}) = \left(\frac{\partial}{\partial \tilde{z}}, \frac{\partial}{\partial \tilde{y}}\right)$. From the second equation of (5.13) with $\tilde{\mu} = 0$, we deduce that the structure functions μ of (A, B) satisfies

$$\mu\beta + \mathrm{L}_A(\beta) = 0 \quad \Longrightarrow \quad \mu + \mathrm{L}_A(\beta)\,\beta^{-1} = 0.$$

By Lemma 3.3 of Chapter 3, we conclude that $\mu + \frac{1}{2\lambda}\mathrm{L}_A(\lambda)\,\mathrm{Id}_m \in \mathrm{Lie}\,(O(p,q))$. It is known that any element $M \in \mathrm{Lie}\,(O(p,q))$ satisfies $\mathrm{diag}\,(M) = 0$, thus we immediately deduce the conditions of (5.19). Moreover, $\Xi'_{p,q}$ is given by a commutative (p,q)-frame (so, in particular, by a pseudo-commutative (p,q)-frame) for which we have $\mathtt{g}_{\tilde{B}} = \mathrm{d}\tilde{y}^t \mathtt{I}_{p,q} \mathrm{d}\tilde{y}$, under any reparametrisation (α, β) that transforms the pseudo-commutative (p,q)-frame $(\tilde{A}, \tilde{B})$ of $\Xi'_{p,q}$ into a pseudo-commutative (p,q)-frame $(\bar{A}, \bar{B})$ of $\Xi_{p,q}$ we have $\mathtt{g}_{\bar{B}} = \lambda \mathtt{g}_{\tilde{B}}$. Therefore, $\mathtt{g}_{\bar{B}}$ is conformally flat, implying that its conformal tensor vanishes.
(v)⇒*(iv)*. Consider a (p,q)-frame (A, B) with structure functions μ satisfying (5.19)

and such that g_B has zero conformal tensor. Take any smooth solutions λ and $\beta^\triangle$ of the following first order partial differential equations

$$\frac{1}{2\lambda}\mathrm{L}_A(\lambda) = -\mu_1^1, \quad \text{and} \quad \mathrm{L}_A(\beta^\triangle)\cdot(\beta^\triangle)^{-1} = -\mu^\triangle. \tag{5.20}$$

Notice that if λ is a solution of the first equation, so is $-\lambda$; therefore we can always assume that $\lambda > 0$. Since $\mu^\triangle \in \mathrm{Lie}\,(O(p,q))$, we have $\beta^\triangle \in O(p,q)$ (see Lemma 3.5 of Chapter 3). Therefore, $\beta = \sqrt{\lambda}\beta^\triangle$ belongs to $GO(p,q)$ and satisfies $\beta^t \mathtt{I}_{p,q}\beta = \lambda \mathtt{I}_{p,q}$. Now, we apply the reparametrisation $(\alpha = 0, \beta)$ and construct a new (p,q)-frame $(\tilde{A}, \tilde{B})$. Using relation (5.13) we obtain

$$\begin{aligned}\frac{1}{\lambda}\beta\tilde{\mu}\beta^{-1} &= \mu + \mathrm{L}_A(\beta)\,\beta^{-1} = \mu^\triangle + \mu_1^1\mathrm{Id}_m + \mathrm{L}_A\left(\sqrt{\lambda}\beta^\triangle\right)\frac{1}{\sqrt{\lambda}}(\beta^\triangle)^{-1},\\ &= \mu^\triangle + \mu_1^1\mathrm{Id}_m + \frac{1}{2\lambda}\mathrm{L}_A(\lambda)\,\mathrm{Id}_m + \mathrm{L}_A(\beta^\triangle)\,(\beta^\triangle)^{-1} = 0,\end{aligned}$$

implying that $\tilde{\mu} = 0$. Hence, $(\tilde{A}, \tilde{B})$ is actually a pseudo-commutative (p,q)-frame. Moreover, by assumption the conformal tensor of $\mathsf{g}_{\tilde{B}}$ vanishes, therefore the pseudo-Riemannian metric $\mathsf{g}_{\tilde{B}}$ is conformally flat.
(iv)$\Rightarrow$*(iii)*. Assume that $(\tilde{A}, \tilde{B})$ is a pseudo-commutative (p,q)-frame (that is, $\tilde{\mu} = 0$) with pseudo-Riemannian metric $\mathsf{g}_{\tilde{B}}$ being conformally flat. We show how to construct a reparametrisation β such that the transformed (p,q)-frame $(\bar{A}, \bar{B})$ is an almost-commutative (p,q)-frame (i.e. its structure functions are $\bar{\mu} = \bar{\nu}_k = 0$).

First, observe that any reparametrisation (α, β) satisfying $\mathrm{L}_{\tilde{A}}(\beta) = 0$ leaves invariant $\tilde{\mu} = 0$. Second, notice that due to the last relation of (5.16a') we have $\mathrm{L}_{\tilde{A}}(\tilde{\nu}_k) = 0$. Therefore, our reasoning can be conducted on the manifold $\mathcal{Y} = \pi(\mathcal{X})$ (defined locally) and, amounts to show that there exists a reparametrisation β, satisfying $\mathrm{L}_{\tilde{A}}(\beta) = 0$, such that $(\bar{A}, \bar{B})$ satisfies $\bar{\nu}_k = 0$.

Since $\mathsf{g}_{\tilde{B}}$ is conformally flat, there exists coordinates $\bar{y} = \varphi(\tilde{y})$ such that

$$\mathsf{g}_{\tilde{B}} = \varphi^*\left(\varrho\sum_{i=1}^{m}\mathtt{I}_i^i\,\mathrm{d}\bar{y}_i\otimes\mathrm{d}\bar{y}_i\right),$$

with $\varrho = \varrho(\bar{y}) > 0$. We denote $\frac{\partial}{\partial\bar{y}} = \left(\frac{\partial}{\partial\bar{y}_1}, \ldots, \frac{\partial}{\partial\bar{y}_m}\right)$ and $\bar{B}_* = (\pi_*\bar{B}_1, \ldots, \pi_*\bar{B}_m)$, where $\bar{B}_i = \varphi_*\tilde{B}_i$. In those coordinates, both $\frac{1}{\sqrt{\varrho}}\frac{\partial}{\partial\bar{y}}$ and $\bar{B}_*$ are pseudo-orthonormal frames so they are related by a pseudo-orthonormal rotation: $\frac{\partial}{\partial\bar{y}} = \sqrt{\varrho}\bar{B}_*\beta^\triangle$, with $\beta^\triangle \in C^\infty(\mathcal{Y}, O(p,q))$. Therefore $\left(\alpha = 0, \sqrt{\varrho}\beta^\triangle\right)$ is a valid reparametrisation which transforms $(\tilde{A}, \tilde{B})$ into $(\bar{A}, \bar{B}) = (\varrho\varphi_*\tilde{A}, \frac{\partial}{\partial\bar{y}} \mod \mathcal{A})$ satisfying $\bar{\mu} = 0$, since $\mathrm{L}_{\tilde{A}}\left(\sqrt{\varrho}\beta^\triangle\right) = 0$, and $\bar{\nu}_k = 0$.
(iii)$\Rightarrow$*(ii)*. Assume that $(\bar{A}, \bar{B})$ is an almost commutative (p,q)-frame (recall that $\bar{\mu} = 0$ and $\bar{\nu}_k = 0$). Applying the reparametrisation $(\alpha, \beta = \mathrm{Id})$ to $(\bar{A}, \bar{B})$, where α is any smooth solution of the equation

$$2\mathrm{L}_A\left(\alpha^t\right) = -\mu^0\mathtt{I}_{p,q},$$

we obtain a new (p,q)-frame $(\hat{A}, \hat{B})$ with $\hat{\mu} = 0$, $\hat{\nu}_k = 0$, and $\hat{\mu}^0 = 0$. It remains to normalise the structure functions $\hat{\nu}_{k,j}^0$, which using the first equation of (5.16a) satisfy $\mathrm{L}_{\hat{A}}\left(\hat{\nu}_{k,j}^0\right) = 0$. In the cotangent bundle of $\mathcal{Y} = \pi(\mathcal{X})$, we introduce $(\theta^1, \ldots, \theta^m)$ the dual frame of $\left(\pi_*\hat{B}_1, \ldots, \pi_*\hat{B}_m\right)$, i.e. $\theta^i(\pi_*\hat{B}_j) = \delta^i_j$. We set $\hat{\nu}^0 = \sum_{k<j} \hat{\nu}^0_{k,j}\theta^k \wedge \theta^j$, then the first equation of (5.16b) implies that $\mathrm{d}\hat{\nu}^0 = 0$, thus by Poincaré lemma (see e.g. [War10, p. 156]) there exist smooth functions $\alpha \in C^\infty(\mathcal{Y}, \mathbb{R}^m)$ satisfying $-2\mathrm{d}\left(\sum_{j=1}^m \alpha^j \mathtt{I}^j_j \theta^j\right) = \hat{\nu}^0$. Expending that equation, we deduce that α fulfils the following set of first order partial differential equation:

$$\mathrm{L}_{\hat{A}}(\alpha) = 0, \quad -2\left(\mathrm{L}_{\hat{B}_k}\left(\alpha^j \mathtt{I}^j_j\right) - \mathrm{L}_{\hat{B}_j}\left(\alpha^k \mathtt{I}^k_k\right)\right) = \hat{\nu}^0_{k,j}, \quad \forall\, 1 \leq k < j \leq m.$$

Applying the reparametrisation given by such α and $\beta = \mathrm{Id}$, we construct a new (p,q)-frame (A, B) whose structure functions satisfy $\mu = \nu = \mu^0 = 0$ and, additionally, using (5.14)

$$\nu^0_{kj} = \hat{\nu}^0_{kj} + 2\mathrm{L}_{\hat{B}_k}\left(\alpha^j \mathtt{I}^j_j\right) - 2\mathrm{L}_{\hat{B}_j}\left(\alpha^k \mathtt{I}^k_k\right) = 0,$$

i.e. (A, B) is a commutative (p,q)-frame.
(ii)$\Rightarrow$*(i)*. Consider a (p,q)-system $\Xi_{p,q}$ such that its (p,q)-frame (A, B) is commutative. Apply a diffeomorphism $(z, y) = \phi(x)$ satisfying $\phi_* A = \frac{\partial}{\partial z}$ and $\phi_* B_i = \frac{\partial}{\partial y_i}$. In those coordinates, $\Xi_{p,q}$ takes the form $\Xi'_{p,q}$. ■

Observe that the conditions of (5.19) can explicitly be checked via the following relation

$$(\mu^\triangle)^t \mathtt{I}_{p,q} + \mathtt{I}_{p,q}\mu^\triangle = 0, \tag{5.21}$$

which describes the elements of $\mathrm{Lie}(O(p,q))$. In the following remarks, we give an interpretation of the conditions of (5.19) and we compare the results of Theorem 5.5 with the ones obtained for lower dimensions in Chapters 2 and 4.

Remark (Summary of the construction of a commutative (p,q)-frame). Under the conditions of statement *(v)*, the proof *(v)*$\Rightarrow$*(ii)*, i.e. the construction of a commutative (p,q)-frame, consists of successively building a feedback (α, β), with $\beta(\cdot) \in GO(p,q)$, and solving the following systems of first order partial differential equations, which can be deduced from equations (5.13) and (5.14), with $\tilde{\mu} = \tilde{\nu}_k = \tilde{\mu}^0 = \tilde{\nu}^0_{k,j} = 0$, by a straightforward but delicate calculation:

$$\mu\beta + \mathrm{L}_A(\beta) = 0, \tag{5.22a}$$

$$\mathrm{L}_B\left(\beta^i_j\right)\beta_k - \mathrm{L}_B\left(\beta^i_k\right)\beta_j + \sum_{s,r=1}^m \beta^s_k \nu^i_{s,r}\beta^r_j = 0, \tag{5.22b}$$

$$2\mathrm{L}_A\left(\alpha^t\right)\mathtt{I}_{p,q} + 2\alpha^t\mu^t\mathtt{I}_{p,q} + \mu^0 - \frac{1}{\lambda}\mathrm{L}_B(\lambda) = 0, \tag{5.22c}$$

$$\begin{aligned} 4\mathtt{I}^k_k\mathtt{I}^j_j&\left(\alpha^k\mathrm{L}_A\left(\alpha^j\right) - \alpha^j\mathrm{L}_A\left(\alpha^k\right)\right) + 2\mathtt{I}^j_j\mathrm{L}_{B_k}\left(\alpha^j\right) - 2\mathtt{I}^k_k\mathrm{L}_{B_j}\left(\alpha^k\right) \\ &+ 2\alpha^k\mathtt{I}^k_k\mu^0_j - 2\alpha^j\mathtt{I}^j_j\mu^0_k + 2\alpha^t\mathtt{I}_{p,q}\left(2\alpha^j\mathtt{I}^j_j\mu_k - 2\alpha^k\mathtt{I}^k_k\mu_j - \nu_{k,j}\right) + \nu^0_{k,j} = 0. \end{aligned} \tag{5.22d}$$

In order to deduce equation (5.22c), we used the following relation: $\mathtt{I}_{p,q}(\mu - 2\mu^1_1\mathrm{Id}_m) = -\mu^t\mathtt{I}_{p,q}$, which follows from equation (5.21). The equations defined by (5.22a) can always be solved, however it is condition (5.19) that guarantees that the solutions β belongs to $C^\infty(\mathcal{X}, GO(p,q))$. To solve (5.22a), we actually solve the two systems given for λ and $\beta^\triangle$ by (5.20) and from those solutions we construct $\beta(\cdot) \in GO(p,q)$ and we build a pseudo-commutative (p,q)-frame. Equation (5.22b) is neither directly given in the proof nor solved there: to construct an almost-commutative frame, we use the conformal flatness of $\mathtt{g}_B$ only. Therefore, the integrability conditions for system (5.22b) must be guaranteed by vanishing of the conformal tensor. Next, we solve equation (5.22c) for α and construct a frame satisfying $\mu^0 = 0$, that frame permits the final step to be performed in the manifold $\mathcal{Y} = \pi(\mathcal{X})$. Finally, we solve (5.22d) for α whose integrability conditions are guaranteed by equation (5.16a) and the first equation of (5.16b) and both are satisfied because of the Jacobi identity.

Observe that given an almost commutative (p,q)-frame (A,B), by adopting a suitable coordinate system, a solution α of (5.22c) and (5.22d) can be chosen explicitly. Indeed, introducing coordinates (z,y) such that $A = \frac{\partial}{\partial z}$ and $B = b^0(x)\frac{\partial}{\partial z} + \frac{\partial}{\partial y}$, with $b^0 = (b^0_1, \ldots, b^0_m)$, we choose the reparametrisation $\left(\alpha = -\frac{1}{2}b^0\mathtt{I}_{p,q}, \beta = \mathrm{Id}\right)$ which yields

$$\tilde{A} = A = \frac{\partial}{\partial z}, \quad \tilde{B} = 2A\left(-\frac{1}{2}b^0\mathtt{I}_{p,q}\right)\mathtt{I}_{p,q} + B = -b^0A + b^0A + \frac{\partial}{\partial y_i} = \frac{\partial}{\partial y_i},$$

showing that $(\tilde{A}, \tilde{B})$ is a commutative (p,q)-frame whose fields are rectified. ◆

Remark (Comparison with the previous results for $m = 1$ and $m = 2$.).

- When $m = 1$, we showed that it is always possible to construct a commutative frame $(\tilde{A}, \tilde{B})$, see Proposition 2.6 of Chapter 2. This is due to the facts that μ consists of only one element μ^1_1 and that $\mathrm{Lie}(O(1)) = 0$, therefore condition (5.19) is always fulfilled. Moreover, we have only one vector field B, so the $(0,2)$-tensor $\mathtt{g}_B$ can be defined as follows in coordinates, where A is rectified, and with a suitable control w. Assume that $A = \frac{\partial}{\partial z}$ and $B = b^0\frac{\partial}{\partial z} + \frac{\partial}{\partial y}$ then we have $\mathtt{g}_B = \mathrm{d}y^2$ which, obviously, is always conformally flat.

- When $m = 2$, the existence of a commutative frame for p-elliptic systems, corresponding to $(p,q) = (2,0)$, is described in Theorem 4.6 of Chapter 4. The conditions expressed there correspond exactly to our condition (5.19) since

  $$\mathrm{Lie}\,(O(2,0)) = \mathrm{vect}_{\mathbb{R}}\left\{\left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right),\ a \in \mathbb{R}\right\}.$$

  There is no condition on $\mathtt{g}_{\tilde{B}}$ since on 2-dimensional manifold all Riemannian metrics are conformally flat.

  The case of p-hyperbolic systems, corresponding to $(p,q) = (1,1)$, is described in Proposition 4.9 of Chapter 4. We used a slightly different approach, given by $\mathtt{I}_{p,q} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so it is not immediate to make a correspondence between the result of this chapter and the previous conditions. Nevertheless, we see that condition (4.35) corresponds to $\mu^\triangle \in \mathrm{Lie}(O(1,1))$ with

  $$\mathrm{Lie}\,(O(1,1)) = \mathrm{vect}_{\mathbb{R}}\left\{\left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right),\ a \in \mathbb{R}\right\}.$$

  Moreover, there is also no condition on $\mathtt{g}_{\tilde{B}}$ since on 2-dimensional manifold all pseudo-Riemannian metrics are conformally flat.

◆

Observe that any (p,q)-system $\Xi'_{p,q}$ is feedback equivalent to (notice the new upper index «$\backprime$»)

$$\Xi^{\backprime}_{p,q} \;:\; \begin{cases} \dot z &= w^t \mathtt{I}_{p,q} w + b_0(x) w + c_0(x) \\ \dot y &= w \end{cases},$$

for which (A,B) is an almost commutative (p,q)-frame satisfying, additionally, $A \wedge C = 0$. In the remaining part of this subsection, we will fully characterise the following normal forms of (p,q)-paraboloid systems (special subclasses of $\Xi'_{p,q}$):

$$\Xi''_{p,q} \;:\; \dot x = w^t \mathtt{I}_{p,q} w \frac{\partial}{\partial z} + \sum_{i=1}^{m} w_i \frac{\partial}{\partial y_i} + c_0(x) \frac{\partial}{\partial z},$$

$$\Xi'''_{p,q} \;:\; \dot x = w^t \mathtt{I}_{p,q} w \frac{\partial}{\partial z} + \sum_{i=1}^{m} w_i \frac{\partial}{\partial y_i} + c_0 \frac{\partial}{\partial z}, \quad c_0 \in \mathbb{R}.$$

Moreover, we will show that the second form can always be normalised into one of three canonical forms with either $c_0 = 0$ or $c_0 = \pm 1$.

The normal form $\Xi''_{p,q}$ describes the intersection between the two equivalent forms $\Xi'_{p,q}$ and $\Xi^{\backprime}_{p,q}$, that is describes the existence of a commutative (p,q)-frame which, additionally, satisfies $A \wedge C = 0$. This last condition fixes $\alpha = 0$ in the reparametrisations and identifies uniquely C, while A is given up to multiplication by λ, and B is given up to a transformation $B \mapsto B\beta$ with $\beta(\cdot) \in GO(p,q)$; so the equivalence of $\Xi_{p,q}$ to $\Xi''_{p,q}$ requires the existence of a reparametrisation (α, β) such that the transformed (p,q)-frame (A,B) is commutative and satisfies $A \wedge C = 0$.

The normal form $\Xi'''_{p,q}$ describes the subclass of $\Xi''_{p,q}$ for which the vector field C is constant, and thus describes (p,q)-paraboloid systems that do not depend on the point $x \in \mathcal{X}$ (called trivial systems in [Ser09]). That class of systems is described by the existence of a triple (A,B,C) with the following properties: (A,B) is a commutative (p,q)-frame satisfying $A \wedge C = 0$ and, additionally, $[A,C] = [B_i, C] = 0$, for all $1 \le i \le m$.

Our conditions will be expressed for $\Xi_{p,q}$ in terms of the structure functions, and therefore are checkable on any (p,q)-paraboloid system $\Xi_{p,q}$; those conditions will, however, be quite complicated to interpret, hence it will be convenient to give, as a corollary, the same conditions for the system $\Xi'_{p,q}$, that is in terms of a commutative (p,q)-frame. Recall that using $\gamma = (\gamma^1, \ldots, \gamma^m)^t$ we define the function

$$\Gamma = \gamma^0 + \gamma^t \mathtt{I}_{p,q} \gamma,$$

on which diffeomorphisms act by composition, furthermore Γ is transformed by $\lambda\tilde{\Gamma} = \Gamma$ under reparametrisations (α, β), as it can be concluded from relation (5.15). Recall, see Chapter 1, that $\mathrm{Riem}\,(\mathrm{g}_B)$ stands for the Riemann curvature tensor and that $\owedge$ stands for the Kulkarni-Nomizu product between symmetric $(0,2)$-tensors.

Theorem 5.6 (Classification results of quadratic systems)**.** *Consider a (p,q)-system $\Xi_{p,q} = (A,B,C)$ with structure functions $\{\mu^0_j, \mu^i_j, \nu^0_{k,j}, \nu^i_{k,j}, \gamma^0, \gamma^i\}$ and let $(\bar A, \bar B)$ be (if it exists) any pseudo-commutative (p,q)-frame equivalent to (A,B). Then, we have*

(i) $\Xi_{p,q}$ is equivalent to $\Xi'_{p,q} = (A', B', C')$ if and only if there exists a reparametrisation (A', B') of (A,B) such that (A', B') is a commutative (p,q)-frame; equivalently, we have

$$\text{(5.23)} \qquad \mu^i_i = \mu^j_j, \quad \mu^{\triangle} \in \mathit{Lie}\,(O(p,q))\,, \quad \text{and} \quad \mathrm{g}_{\bar B} \text{ is conformally flat.}$$

(ii) $\Xi_{p,q}$ *is equivalent to* $\Xi''_{p,q} = (A'', B'', C'')$ *if and only if there exists a reparametrisation* (A'', B'') *of* (A, B) *such that* (A'', B'') *is a commutative* (p,q)*-frame and, additionally,* $A'' \wedge C'' = 0$*; equivalently, if and only if relation (5.23) holds and, additionally, we have*

$$\text{(5.24)}\quad 4I_k^k I_j^j \left(\gamma^k \mathrm{L}_A\left(\gamma^j\right) - \gamma^j \mathrm{L}_A\left(\gamma^k\right)\right) + 2I_k^k \mathrm{L}_{B_j}\left(\gamma^k\right) - 2I_j^j \mathrm{L}_{B_k}\left(\gamma^j\right) = \\ -\nu_{k,j}^0 + 2\gamma^t I_{p,q}\left(2\gamma^k I_k^k \mu_j - 2\gamma^j I_j^j \mu_k - \nu_{k,j}\right) + 2\gamma^k I_k^k \mu_j^0 - 2\gamma^j I_j^j \mu_k^0,$$

$$\text{(5.25)}\quad 2\mathrm{L}_A^2\left(\gamma^t\right) I_{p,q} = \mathrm{L}_A\left(\mu^0 - 2\gamma^t \mu^t I_{p,q}\right) + \mathrm{L}_B\left(\mu_1^1\right) - \mu^0 \mu \\ + 2\mu_1^1 \mu^0 + 2\left(\mathrm{L}_A\left(\gamma^t\right) + \gamma^t \mu^t\right) I_{p,q}\mu,$$

$$\text{(5.26)}\quad 2\mathrm{L}_{B_j}\left(\mathrm{L}_A\left(\gamma^k\right)\right) I_k^k - 2\mathrm{L}_{B_k}\left(\mathrm{L}_A\left(\gamma^j\right)\right) I_j^j = \mathrm{L}_{B_j}\left((\mu^0 - 2\gamma^t \mu^t I_{p,q})_k\right) \\ - \mathrm{L}_{B_k}\left((\mu^0 - 2\gamma^t \mu^t I_{p,q})_j\right) - 2\mu_1^1 \nu_k^0 + \left(\mu^0 - 2\mathrm{L}_A\left(\gamma^t\right) I_{p,q} - 2\gamma^t \mu^t I_{p,q}\right)\nu_{k,j},$$

$$\text{(5.27)}\quad \mathrm{Riem}\left(g_{\bar B}\right) = g_{\bar B} \owedge \left(\sum_{j,k}\left(\mathrm{L}_{B_j}\left(G_k\right) - \sum_{i=1}^m \Gamma_{jk}^i G_i\right)\theta^j \otimes \theta^k - G^t G \right. \\ \left. + \frac{1}{2}\|G\|^2 g_{\bar B}\right),$$

where $G = \frac{1}{2}\mu^0 - \gamma^t \mu^t I_{p,q} - \mathrm{L}_A\left(\gamma^t\right) I_{p,q}$, $\Gamma_{jk}^i = \frac{1}{2} I_i^i\left(\nu_{ij}^k I_k^k + \nu_{ik}^j I_j^j - \nu_{jk}^i I_i^i\right)$, $\theta^j = \pi_*(B_j)^\flat$*, and* $\|\cdot\|^2 = g_{\bar B}(\cdot,\cdot)$.

(iii) $\Xi_{p,q}$ *is equivalent to* $\Xi'''_{p,q} = (A''', B''', C''')$ *if and only if there exists a reparametrisation* (A''', B''') *of* (A, B) *such that* (A''', B''') *is a commutative* (p,q)*-frame satisfying* $A''' \wedge C''' = 0$ *and, additionally,* $[A''', C'''] = [B_i''', C'''] = 0$*; equivalently, if and only if relations (5.23), (5.24), (5.25), (5.26), (5.27) hold and, additionally, we have*

$$\text{(5.28)}\quad \begin{aligned} \mathrm{L}_A\left(\Gamma\right) + 2\Gamma\mu_1^1 &= 0, \\ \mathrm{L}_B\left(\Gamma\right) + 2\Gamma \mathrm{L}_A\left(\gamma^t\right) I_{p,q} - \Gamma\left(\mu^0 - 2\gamma^t \mu^t I_{p,q}\right) &= 0. \end{aligned}$$

Remark (Idea behind the theorem). The idea behind statement *(ii)* of the above theorem is the following. For $\Xi''_{p,q}$, with structure function $\{\tilde\mu_j^0, \tilde\mu_j^i, \tilde\nu_{k,j}^0, \tilde\nu_{k,j}^i, \tilde\gamma^0, \tilde\gamma^i\}$, we have $\tilde\mu^0 = \tilde\mu = \tilde\nu_{k,j}^0 = \tilde\nu_k = 0$, i.e. it has a commutative (p,q)-frame, and $\tilde\gamma^i = 0$. To obtain the last condition, relation (5.15) imposes that $\alpha^i = -\gamma^i$. Therefore, α is fixed and the group of reparametrisations now depends arbitrarily on $\beta \in C^\infty(\mathcal{X}, GO(p,q))$ only. Conditions (5.23), (5.24), (5.25), (5.26), and (5.27) then describe the existence of a reparametrisation β such that a commutative (p,q)-frame exists.

The idea behind statement *(iii)* is generally the same, the additional condition (5.28) ensures that the resulting function c_0 of $\Xi''_{p,q}$ is constant. Indeed, for the system $\Xi''_{p,q}$ we have $\Gamma = c_0$ and the relations of (5.28) imply that $\frac{\partial c_0}{\partial z} = \frac{\partial c_0}{\partial y_i} = 0$, i.e. c_0 is constant. ◆

Proof.

(i) It is actually Theorem 5.5.

(ii) Assume that $\Xi_{p,q}$, with structure functions $\{\mu^0, \mu, \nu_k^0, \nu_k, \gamma^0, \gamma\}$, is feedback equivalent to $\Xi''_{p,q}$, with structure functions $\{\tilde{\mu}^0, \tilde{\mu}, \tilde{\nu}_k^0, \tilde{\nu}_k, \tilde{\gamma}^0, \tilde{\gamma}\}$ satisfying $\tilde{\mu}^0 = \tilde{\mu} = \tilde{\nu}_k^0 = \tilde{\nu}_k = \tilde{\gamma} = 0$, and $\tilde{\gamma}^0 = c_0 \in C^\infty(\mathcal{X})$. The necessity of (5.23) is immediate from Theorem 5.5, since the systems $\Xi''_{p,q}$ form a subclass of $\Xi'_{p,q}$. Using relation (5.15) with $\tilde{\gamma} = 0$, we deduce that $\alpha = -\gamma$. Now we rewrite systems (5.22c) and (5.22d) replacing α by $-\gamma$:

$$\text{(5.22c')} \qquad \frac{1}{\lambda}\mathrm{L}_B(\lambda) = \mu^0 - 2\gamma^t\mu^t\mathtt{I}_{p,q} - 2\mathrm{L}_A(\gamma^t)\,\mathtt{I}_{p,q},$$

$$\text{(5.22d')} \quad 4\mathtt{I}_k^k\mathtt{I}_j^j\left(\gamma^k\mathrm{L}_A(\gamma^j) - \gamma^j\mathrm{L}_A(\gamma^k)\right) + 2\mathtt{I}_k^k\mathrm{L}_{B_j}(\gamma^k) - 2\mathtt{I}_j^j\mathrm{L}_{B_k}(\gamma^j)$$
$$+ 2\gamma^j\mathtt{I}_j^j\mu_k^0 - 2\gamma^k\mathtt{I}_k^k\mu_j^0 - 2\gamma^t\mathtt{I}_{p,q}\left(2\gamma^k\mathtt{I}_k^k\mu_j - 2\gamma^j\mathtt{I}_j^j\mu_k - \nu_{k,j}\right) + \nu_{k,j}^0 = 0.$$

Equation (5.22d') is exactly condition (5.24). Moreover, the integrability conditions of system (5.22c') together with the equation $\frac{1}{\lambda}\mathrm{L}_A(\lambda) = -2\mu_1^1$ give conditions (5.25) and (5.26). Finally, using that $\mathsf{g}_{\tilde{B}}$ is flat (as it can be seen in a suitable coordinate system) and that $\mathsf{g}_{\tilde{B}} = \lambda\mathsf{g}_{\bar{B}}$ we conclude, using relation (1.3) of Chapter 1 with $\phi = \ln\sqrt{\lambda}$, that

$$\mathrm{Riem}(\mathsf{g}_B) = \mathsf{g}_B \owedge \left(\mathrm{Hess}(\phi) - \mathrm{d}\phi\otimes\mathrm{d}\phi + \frac{1}{2}\|\mathrm{grad}(\phi)\|^2\mathsf{g}_B\right).$$

Next it is a straightforward but painful calculation to express $\mathrm{Hess}\left(\ln\sqrt{\lambda}\right)$, $\mathrm{grad}\left(\ln\sqrt{\lambda}\right)$, and $\mathrm{d}\left(\ln\sqrt{\lambda}\right)$, using structure functions. Conversely, assume that the structure functions $\{\mu^0, \mu, \nu_k^0, \nu_k, \gamma^0, \gamma\}$ of $\Xi_{p,q}$ satisfy (5.23), (5.24), (5.25), (5.26), and (5.27), then by Theorem 5.5 $\Xi_{p,q}$ can be transformed into $\Xi'_{p,q}$ for which our conditions (5.25) and (5.26) read

$$\mathrm{L}_A^2(\gamma) = 0, \quad \text{and} \quad \mathrm{L}_{B_j}\left(\mathrm{L}_A(\gamma^k)\right)\mathtt{I}_k^k - \mathrm{L}_{B_k}\left(\mathrm{L}_A(\gamma^j)\right)\mathtt{I}_j^j = 0.$$

Those relations are the integrability conditions for the following system of first order linear partial differential equations:

$$\mathrm{L}_A(\lambda) = 0, \quad \mathrm{L}_B(\lambda) = -2\lambda\mathrm{L}_A(\gamma^t)\,\mathtt{I}_{p,q},$$

which therefore admit a smooth solution λ. Notice that to ensure $\lambda \neq 0$ we may actually solve the above system for $\ln(\lambda)$ instead. We apply the reparametrisation $\left(\alpha = -\gamma, \beta = \sqrt{\lambda}\mathrm{Id}_m\right)$ and in a coordinate system $(\tilde{z}, \tilde{y})$, where (A, B) is rectified, $\Xi'_{p,q}$ takes the form

$$\begin{cases} \dot{\tilde{z}} &= \lambda\tilde{w}^t\mathtt{I}_{p,q}\tilde{w} - 2\sqrt{\lambda}\gamma^t\mathtt{I}_{p,q}\tilde{w} + \Gamma \\ \dot{\tilde{y}} &= \sqrt{\lambda}\tilde{w} \end{cases}.$$

Condition (5.24) ensures that there exists a function $z(\tilde{z}, \tilde{y})$ satisfying $\frac{\partial z}{\partial \tilde{z}}\lambda = 1$ and $\frac{\partial z}{\partial \tilde{y}} = \frac{2}{\lambda}\gamma^t\mathtt{I}_{p,q}$. Using z as a new coordinate yields

$$\text{(5.30)} \qquad \begin{cases} \dot{z} &= \tilde{w}^t\mathtt{I}_{p,q}\tilde{w} + \Gamma \\ \dot{\tilde{y}} &= \sqrt{\lambda}\tilde{w} \end{cases}.$$

Finally, condition (5.27) implies that $\mathsf{g}_B = \frac{1}{\lambda} \mathrm{d}\tilde{y}^t \mathtt{I}_{p,q} \mathrm{d}\tilde{y}$ is flat. Therefore, there exists coordinates $y = \psi(\tilde{y})$ such that $\mathsf{g}_B = \psi^*(\mathrm{d}y^t \mathtt{I}_{p,q} \mathrm{d}y)$. Hence $B_i = \sqrt{\lambda}\frac{\partial}{\partial \tilde{y}_i}$ and the vector fields $\frac{\partial}{\partial y_i}$ form a pseudo-orthonormal frame, so they differ by a pseudo orthonormal rotation $\beta^{\triangle}$. After applying this rotation, we obtain the form $\Xi''_{p,q}$ in the coordinates (z, y).

(iii) Assume that $\Xi_{p,q}$, with structure functions $\{\mu^0, \mu, \nu_k^0, \nu_k, \gamma^0, \gamma\}$, is feedback equivalent to $\Xi'''_{p,q}$, with structure functions $\{\tilde{\mu}^0, \tilde{\mu}, \tilde{\nu}_k^0, \tilde{\nu}_k, \tilde{\gamma}^0, \tilde{\gamma}\}$ satisfying $\tilde{\mu}^0 = \tilde{\mu} = \tilde{\nu}_k^0 = \tilde{\nu}_k = \tilde{\gamma} = 0$, and $\tilde{\gamma}^0 = c_0 \in \mathbb{R}$. The necessity of conditions (5.23), (5.24), (5.25), (5.26), and (5.27) is clear from the previous item of the proof and we show that (5.28) is necessary. For $\Xi'''_{p,q}$ we have $\tilde{\Gamma} = c_0 \in \mathbb{R}$, and under reparametrisations we have $\tilde{\Gamma} = \frac{\Gamma}{\lambda}$, where $\Gamma = \gamma^0 + \gamma^t \mathtt{I}_{p,q} \gamma$. We differentiate the last relation along $\tilde{A}$ and $\tilde{B}$:

$$\mathrm{L}_{\tilde{A}}\left(\tilde{\Gamma}\right) = 0 = \lambda \mathrm{L}_A\left(\frac{\Gamma}{\lambda}\right) = \frac{1}{\lambda}\left(\mathrm{L}_A(\Gamma)\lambda - \Gamma \mathrm{L}_A(\lambda)\right) = \frac{1}{\lambda}\left(\mathrm{L}_A(\Gamma)\lambda + 2\Gamma\lambda\mu_1^1\right),$$

$$\mathrm{L}_{\tilde{B}}\left(\tilde{\Gamma}\right) = 0 = 2\mathrm{L}_{A\alpha^t \mathtt{I}_{p,q}\beta}\left(\frac{\Gamma}{\lambda}\right) + \mathrm{L}_B\left(\frac{\Gamma}{\lambda}\right)\beta = 0 + \frac{1}{\lambda^2}\left(\mathrm{L}_B(\Gamma)\lambda\beta - \Gamma \mathrm{L}_B(\lambda)\beta\right),$$

$$\Rightarrow \quad 0 = \mathrm{L}_B(\Gamma)\lambda\beta - \Gamma\left(\lambda\mu^0\beta + 2\lambda \mathrm{L}_A\left(-\gamma^t \mathtt{I}_{p,q}\beta\right) + 2\gamma \mathtt{I}_{p,q}\beta \mathrm{L}_A(\lambda)\right),$$

$$0 = \mathrm{L}_B(\Gamma) - \Gamma\left(\mu^0 - 2\mathrm{L}_A\left(\gamma^t\right)\mathtt{I}_{p,q} + 2\gamma^t \mathtt{I}_{p,q}\mu - 4\gamma^t \mathtt{I}_{p,q}\mu_1^1\right),$$

where we used the relations (5.22c') and $\mathrm{L}_A(\lambda) = -2\lambda\mu_1^1$. Conversely, assume that $\Xi_{p,q}$ satisfies (5.23), (5.24), (5.25), (5.26), (5.27), and (5.28). Then, by statement *(ii)*, $\Xi_{p,q}$ can be brought into the form $\Xi''_{p,q}$ for which we have $(A, B) = \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial y}\right)$ and $\Gamma = c_0(x)$, thus condition (5.28) implies $\frac{\partial c_0}{\partial z} = \frac{\partial c_0}{\partial y_i} = 0$ and, finally, $c_0 \in \mathbb{R}$, i.e. we indeed have the normal form $\Xi'''_{p,q}$.

■

Remark (Role of the conditions). Despite the complexity of their expression, the role of each condition of the above theorem is clear. First, condition (5.23) implies that there exists a (non-unique) commutative (p, q)-frame. Second, conditions (5.25) and (5.26) are used to construct a function λ that rescale the (p, q)-frame (but à-priori we lose commutativity). Third, condition (5.24) implies the existence of a coordinate system in which $A = \frac{\partial}{\partial z}$ and the fields B_i are expressed by $\sqrt{\lambda}\frac{\partial}{\partial y_i}$. At this point we get the normal form (5.30). Finally, we can act on this normal form only with reparametrisation $w = \beta\tilde{w}$ with $\beta \in C^\infty(\mathcal{Y}, O(p, q))$. Therefore, in order to obtain the form $\Xi''_{p,q}$ we need to rectify the pseudo-Riemannian metric $\mathsf{g}_B = \frac{1}{\lambda}\mathrm{d}y^t \mathtt{I}_{p,q} \mathrm{d}y$ on the classical pseudo-Euclidean one, which is possible if and only if its Riemannian curvature tensor vanishes. So condition (5.27) ensures that the previous rescaling by λ produces a flat metric.

Finally, condition (5.28) implies that the function $c_0(x)$ is actually constant and thus produces the form $\Xi'''_{p,q}$. ◆

As announced, we now give the conditions of the previous theorem in a commutative (p, q)-frame so a lot of coefficients vanish and thus it will be easier to interpret them.

Corollary 5.1 (Classification of $\Xi'_{p,q}$). *Consider a (p, q)-system $\Xi'_{p,q} = (A, B, C)$ with structure functions $(\mu, \mu^0, \nu_k, \nu_k^0, \gamma^0, \gamma) = (0, 0, 0, 0, \gamma^0, \gamma)$, then we have* *(i) $\Xi'_{p,q}$ is equivalent to $\Xi''_{p,q}$ if and only if we have*

$$(5.24') \qquad 4 I_k^k I_j^j \left(\gamma^k \mathrm{L}_A\left(\gamma^j\right) - \gamma^j \mathrm{L}_A\left(\gamma^k\right)\right) + 2 I_k^k \mathrm{L}_{B_j}\left(\gamma^k\right) - 2 I_j^j \mathrm{L}_{B_k}\left(\gamma^j\right) = 0$$

$$(5.25') \qquad \mathrm{L}_A^2\left(\gamma\right) = 0,$$

$$(5.26') \qquad \mathrm{L}_A\left(\mathrm{L}_{B_j}\left(\gamma^k\right) I_k^k - \mathrm{L}_{B_k}\left(\gamma^j\right) I_j^j\right) = 0,$$

$$(5.27') \quad g_B \owedge \left(-\sum_{j,k}\left(\mathrm{L}_{B_j}\left(\mathrm{L}_A\left(\gamma^k\right)\right) I_k^k + \frac{1}{2}\mathrm{L}_A^2\left(\gamma^k\gamma^j\right) I_k^k I_j^j\right)\theta^j \otimes \theta^k + \frac{1}{2}\|\mathrm{L}_A\left(\gamma\right)\|^2 g_B\right) = 0.$$

(ii) $\Xi'_{p,q}$ is equivalent to $\Xi'''_{p,q}$ if and only if (5.24'), (5.25'), (5.26'), (5.27') hold and, additionally, we have

$$(5.28') \qquad \mathrm{L}_A\left(\Gamma\right) = \mathrm{L}_B\left(\Gamma\right) + 2\Gamma \mathrm{L}_A\left(\gamma^t\right) I_{p,q} = 0.$$

Our conditions are generalisations of the conditions obtained in the case $m = 2$, compare Corollaries 4.1 and 4.3 of Chapter 4. The only difference is that we were able to work out a simpler form of condition (5.27').

Remark (Interpretation of the conditions). Consider a (p,q)-system $\Xi'_{p,q}$ together with its commutative (p,q)-frame $(A,B) = \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial y}\right)$ and with structure functions $\mu^0 = \mu = \nu_k^0 = \nu_k = 0$. Assume that $\Xi'_{p,q}$ is equivalent to $\Xi'''_{p,q}$. Then condition (5.25) implies that we have

$$\gamma^i(z,y) = \gamma_0^i(y)z + \gamma_1^i(y).$$

By integrating the equations of (5.28) we deduce that

$$\Gamma(y) = G\exp\left(-2\int\sum_{i=1}^{m}\gamma_0^i \mathrm{I}_i^i \,\mathrm{d}y\right), \quad G \in \mathbb{R}.$$

Therefore the systems $\Xi'_{p,q}$ that are equivalent to $\Xi'''_{p,q}$ are parametrised by $2m$ functions of y (which satisfy additional constraints) and a real constant. The sign of that constant gives rise to canonical forms, see below. ◆

The following proposition gives a canonical from of systems $\Xi'''_{p,q}$ depending on either $c_0 \neq 0$ of $c_0 = 0$.

Proposition 5.5 (Canonical form of $\Xi'''_{p,q}$)**.** *Consider a (p,q)-system $\Xi_{p,q}$ with structure functions $\{\mu^0, \mu, \nu^0, \nu, \gamma^0, \gamma\}$ satisfying (5.24), (5.25), (5.26), (5.27), and (5.28). Then it always admits one of the following canonical form*

$$\Xi^0_{p,q} : \dot{x} = w^t I_{p,q} w \frac{\partial}{\partial z} + \sum_{i=1}^{m} w_i \frac{\partial}{\partial y_i} + 0, \quad \textit{or}$$

$$\Xi^\varepsilon_{p,q} : \dot{x} = w^t I_{p,q} w \frac{\partial}{\partial z} + \sum_{i=1}^{m} w_i \frac{\partial}{\partial y_i} + \varepsilon_{p,q}\frac{\partial}{\partial z},$$

with $\varepsilon_{p,q} = \begin{cases} \pm 1 & \text{if} \quad p \neq q \\ 1 & \text{if} \quad p = q \end{cases}$. *Moreover,* $\Xi_{p,q}$ *is equivalent to the former if and only if* $\Gamma \equiv 0$ *and to the latter if and only if* $\Gamma > 0$ *or* $\Gamma < 0$ *when* $p \neq q$*, or* $\Gamma \neq 0$ *when* $p = q$*.*

Compare those canonical forms with the ones obtained in the cases $m = 1$ and $m = 2$, see Theorem 2.4 of Chapter 2 and Propositions 4.5 and 4.10 of Chapter 4.

Proof. Assume that $\Xi_{p,q}$ satisfies (5.24), (5.25), (5.26), (5.27), and (5.28), then it is equivalent to $\Xi'''_{p,q}$. If $c_0 = 0$, then we have the canonical form $\Xi^0_{p,q}$. Otherwise, we use the following feedback transformation

$$\tilde{z} = \frac{z}{|c_0|}, \quad \tilde{y}_i = \frac{y_i}{\sqrt{|c_0|}}, \quad \text{and} \quad \tilde{w}_i = \frac{w_i}{\sqrt{|c_0|}}$$

to obtain $\Xi^{\varepsilon}_{p,q}$ with $\varepsilon_{p,q} = \pm 1$. If $p \neq q$ then we can not further normalise. If $p = q$ then by the coordinate change $\bar{z} = -\tilde{z}$ we can change $\varepsilon_{p,p} = -1$ into $\varepsilon_{p,p} = +1$.

Under reparametrisations (α, β) we always have $\lambda\tilde{\Gamma} = \Gamma$, thus clearly $\Xi_{p,q}$ (satisfying (5.24), (5.25), (5.26), (5.27), and (5.28)) is equivalent to $\Xi^0_{p,q}$ if and only if $\Gamma \equiv 0$. If $\Gamma \neq 0$, then its sign is an invariant in the case $p > q$ because in that case feedback transformations $\beta \in G0(p,q)$ satisfy $\lambda > 0$. If $p = q$ then λ satisfies $\lambda \neq 0$ and the sign of Γ is not invariant. ■

We terminate this subsection by explaining how to get normal and canonical forms of (p,q)-paraboloid submanifolds $\mathcal{S}_Q$. Recall that those submanifolds are given by an equation of the form $\dot{z} = \dot{y}^t Q(x)\dot{y} + b(x)\dot{y} + c(x)$, with $Q = (Q^i_j(x))$ a symmetric matrix of constant signature (p,q) satisfying $\det Q \neq 0$, and $b = (b_1, \ldots, b_m)$. Hence a direct parametrisation of $\mathcal{S}_Q$, in terms of a control system, is given by

$$\Xi_{\mathcal{S}_Q} \; : \; \begin{cases} \dot{z} & = w^t Q(x) w + b(x) w + c(x) \\ \dot{y} & = w \end{cases} .$$

Observe that the above system $\Xi_{\mathcal{S}_Q}$ is not of the previously used form $\Xi_{p,q}$ but by a similar argument as the one used for Lemma 3.1 in Chapter 3, we transform $\Xi_{\mathcal{S}_Q}$ into a system $\Xi^{\mathcal{S}}_{p,q}$ satisfying $A = \frac{\partial}{\partial z}$, the fields B_i depend on the functions Q^i_j and b_i, and $C = c(x)\frac{\partial}{\partial z}$. The conditions of the previous results Proposition 5.4 and Theorem 5.6 can be tested on $\Xi^{\mathcal{S}}_{p,q}$ and the normal forms obtained give normal forms of $\mathcal{S}_Q$. More precisely, we have

Corollary 5.2 (Normal and canonical forms of paraboloid submanifold)**.** *Consider a paraboloid submanifold* $\mathcal{S}_{p,q} = \{\dot{z} = \dot{y}^t Q(x)\dot{y} + b(x)\dot{y} + c(x)\}$ *together with its parametrisation* $\Xi^{\mathcal{S}}_{p,q}$*. The following statements hold:*

(i) If $\Xi^{\mathcal{S}}_{p,q}$ *is equivalent to* $\Xi^d_{p,q}$*, then* $\mathcal{S}_Q$ *is equivalent to*

$$\mathcal{S}^d_Q = \{\dot{y}^t D(x)\dot{y} + b(x)\dot{y} + c(x)\},$$

where $D(x)$ *is a diagonal matrix with signature* (p,q)*.*

(ii) If $\Xi^{\mathcal{S}}_{p,q}$ *is equivalent to* $\Xi'_{p,q}$*, then* $\mathcal{S}_Q$ *is equivalent to*

$$\mathcal{S}'_Q = \{\dot{y}^t I_{p,q}\dot{y} + b(x)\dot{y} + c(x)\}.$$

(iii) If $\Xi^{\mathcal{S}}_{p,q}$ is equivalent to $\Xi''_{p,q}$, then $\mathcal{S}_Q$ is equivalent to $\mathcal{S}''_Q = \{\dot{y}^t\, I_{p,q}\dot{y} + c(x)\}$.

(iv) If $\Xi^{\mathcal{S}}_{p,q}$ is equivalent to $\Xi'''_{p,q}$, then $\mathcal{S}_Q$ is equivalent to $\mathcal{S}'''_Q = \{\dot{y}^t\, I_{p,q}\dot{y} + c\}$, with $c \in \mathbb{R}$, moreover, c can always be normalised to either $c = 0$ or $c = \pm 1$.

Remark. All items are actually «if and only if» statements but we presented them as implications that show how equivalence of control systems allows to solve the original problem of equivalence of paraboloid submanifolds. ◆

The normal form $\mathcal{S}^d_Q$ describes the smooth diagonalisation of Q, therefore our characterisation can be interpreted as a smooth version of the singular value decomposition theorem. For the normal form $\mathcal{S}'_Q$, the matrix Q is fully normalised to the constant matrix $\mathtt{I}_{p,q}$, the conditions imply that a certain metric ($\mathtt{g}_{\bar{B}}$ associated to any pseudo-commutative (p,q)-frame of $\Xi^{\mathcal{S}}_{p,q}$) is conformally flat, justifying to call this class of paraboloid submanifolds *weakly-flat*. For $\mathcal{S}''_Q$ we additionally normalised b; our conditions imply that the Riemannian curvature tensor of the metric $\mathtt{g}_B$ fails vanishing by a conformal transformation, which justifies to call this class of paraboloid submanifold *strongly-flat*. Finally our characterisation of $\mathcal{S}'''_Q$ describes paraboloid submanifolds with no functional parameters, i.e. which do not depend on the point $x \in \mathcal{X}$.

In this subsection, we studied the classification problem of nonlinear (p,q)-paraboloid systems under the action of the group of feedback transformations. Our classification includes several normal forms and canonical forms. The conditions that we introduced are checkable in terms of algebraic and differential relations between structure functions attached to the (p,q)-paraboloid structure of the system. Our classification gives an equivalent classification of several normalisation of paraboloid submanifolds.

3 Conclusion and Perspectives

In this chapter, we presented general results on a characterisation and a classification of paraboloid control systems, which equivalently form general results for characterising and classifying paraboloid submanifolds. Our results generalise those of Chapter 4 (which is devoted to the special case of $\mathcal{X}$ being 3-dimensional), as well as those of Chapter 2 for parabolic systems on 2-dimensional manifolds.

Our characterisation results introduce a new class of control-affine systems called (p,q)-systems, which are second prolongations of paraboloid susbmanifolds. We define the notion of weak and strong quadratic frames of an involutive distribution and we show that the existence of a strong frame is the key of our characterisation of (p,q)-systems. Next, our classification results are based on feedback transformations of control-nonlinear (p,q)-systems (first prolongations of paraboloid submanifolds). To those systems we attach a frame of the tangent bundle, called a (p,q)-frame, and we give necessary and sufficient conditions (both in terms of relations between structure functions attached to (p,q)-frames and also in terms of geometric properties) for the equivalence of a general (p,q)-frame to some specific subclasses. Different classes of (p,q)-frames give different normal forms of (p,q)-systems.

In the future, we plan to study more carefully the conditions of Theorems 5.5 and 5.6 and express them for a general (p,q)-frame. We also want to study and describe more deeply the geometry of our conditions. Afterwards, we would like to consider the problem of equivalence of general quadric submanifolds. Then it would interesting to study the following generalisations. First, one could analyse quadric submanifolds of codimension $k \geq 2$ (i.e. given by the zero level-set of a quadratic map $T\mathcal{X} \to \mathbb{R}^k$), second, one could try to generalise our results to submanifolds given by a polynomial of any degree with respect to the velocities.

Chapter 6

Characterisation of paraboloid systems by their Lie algebra of infinitesimal symmetries

In this chapter, we will study the infinitesimal symmetries of (p,q)-paraboloid control-affine systems. We will show that isomorphisms of the Lie algebra of symmetries are reflected in feedback equivalence of the corresponding control-affine systems. Precisely, we will study the Lie algebra of infinitesimal symmetries of the following null-form of (p,q)-paraboloid systems given by

$$\Sigma^0_{p,q} : \begin{cases} \dot{z} &= w^t \mathtt{I}_{p,q} w \\ \dot{y} &= w \\ \dot{w} &= u \end{cases}, \quad (z,y,w) \in \mathbb{R}^{2m+1}, \quad u \in \mathbb{R}^m,$$

and $\mathtt{I}_{p,q} = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}$. We will show that this class of control-affine systems is determined by its Lie algebra of symmetries.

Recall that all relevant definitions and properties about Lie algebras and infinitesimal symmetries are given in section 1 of Chapter 1. This chapter is organised as follows. In the next section, we present the results obtained for the single-input case, i.e. for parabolic systems. And then, we will give a general result which will contain all systems of the form $\Sigma^0_{p,q}$.

1 Introductory case, single-input paraboloid systems

This section is quoted from the paper [SR21] submitted to *Journal of Dynamical and Control System*. Below, we restrict the results obtained in that paper to the class of parabolic systems only. We also adapt the notation of the paper to that used in this thesis. The interested reader will find in our paper results generalising Theorem 6.1 to the class of elliptic and hyperbolic control systems.

In this section, we give checkable necessary and sufficient conditions for a control-affine system $\Sigma = (f,g)$ to be feedback equivalent to the form

$$\Sigma^0_{1,0} : \begin{cases} \dot{z} &= w^2 \\ \dot{y} &= w \\ \dot{w} &= u \end{cases}, \quad (z,y,w) \in \mathbb{R}^3, \quad u \in \mathbb{R}.$$

This class of systems is called null-form parabolic system because of the absence of parameters. Our approach is based on the study of the Lie algebra of infinitesimal symmetries of Σ, which presents the main advantage of being applicable to any control-affine system whereas the classification approach that we presented in the previous chapters applies to the class of paraboloid systems only (for this class of systems see, in particular, Theorem 2.4 of Chapter 2).

Using Proposition 1.4 of Chapter 1, it is an easy computation to find the Lie algebra of infinitesimal symmetries of the system $\Sigma^0_{1,0}$, which is

$$\mathfrak{L}^0_{1,0} = \mathrm{vect}_{\mathbb{R}} \left\{ \frac{\partial}{\partial z}, \frac{\partial}{\partial y}, 2z\frac{\partial}{\partial z} + y\frac{\partial}{\partial y} + w\frac{\partial}{\partial w} \right\}.$$

This Lie algebra has an abelian Lie ideal $\mathfrak{I} = \mathrm{vect}_{\mathbb{R}} \left\{ \frac{\partial}{\partial z}, \frac{\partial}{\partial y} \right\}$, which corresponds to the fact that the system $\Sigma^0_{1,0}$ is invariant under translations of the form $(z, y) \mapsto (z + c, y + d)$, with $c, d \in \mathbb{R}$. The third vector field of $\mathfrak{L}^0_{1,0}$ plays the role of an Euler vector field: it encodes the relative homogeneity degree of the components of the drift $f = w^2\frac{\partial}{\partial z} + w\frac{\partial}{\partial y}$, namely it means that the component of f along the z-coordinate has twice the homogeneity degree as the component along the y-coordinate. Denoting $v_1 = \frac{\partial}{\partial z}$, $v_2 = \frac{\partial}{\partial y}$, and $v_3 = 2z\frac{\partial}{\partial z} + y\frac{\partial}{\partial y} + w\frac{\partial}{\partial w}$, the commutativity relations of the (above) generators of $\mathfrak{L}^0_{1,0}$ are

$$[v_1, v_2] = 0, \quad [v_1, v_3] = 2v_1, \quad \text{and} \quad [v_2, v_3] = v_2. \tag{6.1}$$

Using those relations, we identify $\mathfrak{L}^0_{1,0}$ in the classification of 3-dimensional Lie algebras as presented by Winternitz in [Bow05]. In that classification, we have $\mathfrak{L}^0_{1,0} \cong L(3, 2, 2)$ with no immediate interpretation as the Lie algebra of a remarkable Lie group. Observe that $\mathfrak{I}$ is equal to the derived algebra of $\mathfrak{L}$, i.e. $\mathfrak{I} = [\mathfrak{L}, \mathfrak{L}]$. The following proposition gives checkable conditions to identify the Lie algebra $\mathfrak{L}^0_{1,0}$ among all 3-dimensional Lie algebras.

Proposition 6.1 (Characterisation of $\mathfrak{L}^0_{1,0}$). *A 3-dimensional Lie algebra $\mathfrak{L}$ is isomorphic to $\mathfrak{L}^0_{1,0}$ if and only if*

(i) The ideal $\mathfrak{I} = [\mathfrak{L}, \mathfrak{L}]$ is 2-dimensional and abelian,

(ii) For any element $l \in \mathfrak{L}/\mathfrak{I}$, its action $\mathrm{ad}_l : \mathfrak{I} \to \mathfrak{I}$ is diagonalisable with two non-zero eigenvalues λ_1 and λ_2 related by $\lambda_1 = 2\lambda_2$.

Proof. The proof is adapted from [Bow05, Proposition 5. Case 1]. The necessity of the conditions is immediate from the explicit form of the generators of $\mathfrak{L}^0_{1,0}$ and the fact that they don't depend on a particular choice of generators of $\mathfrak{L}^0_{1,0}$. Conversely, assume that $\mathfrak{L}$ is a 3-dimensional Lie algebra satisfying conditions *(i)* and *(ii)*. Let e_1, e_2, and e_3 be three vectors such that $\mathfrak{L} = \mathrm{vect}_{\mathbb{R}} \{e_1, e_2, e_3\}$ and $\mathfrak{I} = \mathrm{vect}_{\mathbb{R}} \{e_1, e_2\}$. The action of e_3 on $\mathfrak{I}$ by ad_{e_3} is diagonalisable, i.e. there exists an invertible linear map $P : \mathfrak{I} \to \mathfrak{I}$ such that $[e_3, v_1] = \lambda_1 v_1$ and $[e_3, v_2] = \lambda_2 v_2$, where $v_1 = Pe_1$ and $v_2 = Pe_2$ form a new frame of $\mathfrak{I}$. We now set $v_3 = -\frac{2}{\lambda_1}e_3$ and, in this new basis, we obtain the following commutativity relations for $\mathfrak{L}$

$$\begin{aligned}
[v_1, v_2] = 0, \quad [v_1, v_3] &= \left[v_1, \frac{-2}{\lambda_1}e_3 \right] = \frac{2}{\lambda_1}[e_3, v_1] = 2v_1, \\
[v_2, v_3] &= \left[v_2, \frac{-2}{\lambda_1}e_3 \right] = \frac{2}{\lambda_1}[e_3, v_2] = v_2,
\end{aligned}$$

recall that we have $\lambda_1 = 2\lambda_2$. Hence, in the frame $\{v_1, v_2, v_3\}$, we get that $\mathfrak{L}$ has the same commutativity relations as $\mathfrak{L}^0_{1,0}$. Therefore, by Proposition 1.3 of Chapter 1, those two Lie algebras are isomorphic. ■

We now formulate and prove that the Lie algebra $\mathfrak{L}$ of Σ, being isomorphic to $\mathfrak{L}^0_{1,0}$, characterises those 3-dimensional control-affine systems Σ (with scalar control, i.e. $m = 1$) that are feedback equivalent to $\Sigma^0_{1,0}$. Recall that we attach to a control system the distributions $\mathcal{D}^0 = \operatorname{span}\{g\}$ and $\mathcal{D}^1 = \operatorname{span}\{g, \operatorname{ad}_f g\}$. Notice that the drift $f = w^2\frac{\partial}{\partial z} + w\frac{\partial}{\partial y}$ of $\Sigma^0_{1,0}$ considered locally around (z_0, y_0, w_0) possesses an equilibrium if $w_0 = 0$ and not if $w_0 \neq 0$. As a consequence, the system $\Sigma^0_{1,0}$ possesses two non-equivalent local normal forms, which can be represented around $w_0 = 0$ by

$$\Sigma^{0,0}_{1,0} : \begin{cases} \dot{z} &= w^2 \\ \dot{y} &= w \\ \dot{w} &= u \end{cases} \quad \text{and} \quad \Sigma^{0,1}_{1,0} : \begin{cases} \dot{z} &= (w+1)^2 \\ \dot{y} &= w+1 \\ \dot{w} &= u \end{cases}, \quad \text{respectively.}$$

Theorem 6.1 ($\Sigma^0_{1,0}$ is characterised by $\mathfrak{L}^0_{1,0}$)**.** *Consider, locally around ξ_0, a control-affine system $\Sigma : \dot{\xi} = f(\xi)+g(\xi)u$ on a 3-dimensional smooth manifold with a scalar control, and let $\mathfrak{L}$ be its Lie algebra of infinitesimal symmetries.*

(i) Σ is locally feedback equivalent to $\Sigma^{0,1}_{1,0}$ around $(z_0, y_0, 0)$ if and only if

$$\mathfrak{L} \cong \mathfrak{L}^0_{1,0}, \quad \mathfrak{I}(\xi_0) \oplus \mathcal{D}^0(\xi_0) = T_{\xi_0}\mathbb{R}^3, \quad \textit{and} \quad f(\xi_0) \notin \mathcal{D}^0(\xi_0);$$

(ii) Σ is locally feedback equivalent to $\Sigma^{0,0}_{1,0}$ around $(z_0, y_0, 0)$ if and only if

$$\mathfrak{L} \cong \mathfrak{L}^0_{1,0}, \quad \mathfrak{I}(\xi_0) \oplus \mathcal{D}^0(\xi_0) = T_{\xi_0}\mathbb{R}^3, \quad f(\xi_0) \in \mathcal{D}^0(\xi_0), \quad \textit{and} \quad \dim \mathcal{D}^1(\xi_0) = 2.$$

In our paper [SR21], we extend the above result to the case of elliptic and hyperbolic control systems. Notice that in statement *(ii)* the condition on the pointwise rank of the distribution $\mathcal{D}^1$ can be replaced by $g \wedge \operatorname{ad}_g f(\xi_0) \neq 0$ (compare this assertion with the assumptions of Proposition 6.2 below).

Proof. We show the sufficiency part of the statements only as their necessity follows immediately from the study of $\Sigma^0_{1,0}$, from its Lie algebra of symmetries $\mathfrak{L}^0_{1,0}$, and from the feedback invariance of our conditions. Consider the system $\Sigma = (f, g)$, given by vector fields f and g, and let three vector fields v_1, v_2, v_3 generate the 3-dimensional Lie algebra $\mathfrak{L} = \operatorname{vect}_{\mathbb{R}}\{v_1, v_2, v_3\}$ of infinitesimal symmetries, which by assumption is isomorphic to $\mathfrak{L}^0_{1,0}$. We can assume that the abelian ideal of $\mathfrak{L}$ is $\mathfrak{I} = \operatorname{vect}_{\mathbb{R}}\{v_1, v_2\}$ and that v_1, v_2, v_3 satisfy the commutativity relations (6.1). Since $\mathfrak{I}(\xi_0) \oplus \mathcal{D}^0(\xi_0) = T_{\xi_0}\mathbb{R}^3$, it follows that v_1, v_2, and g are independent, locally around ξ_0. We apply a local diffeomorphism $\psi(\xi) = (\tilde{z}, \tilde{y}, \tilde{w})$, $\psi(\xi_0) = 0 \in \mathbb{R}^3$, such that $\tilde{v}_1 = \psi_* v_1 = \frac{\partial}{\partial \tilde{z}}$, $\tilde{v}_2 = \psi_* v_2 = \frac{\partial}{\partial \tilde{y}}$, and $\tilde{g} = \psi_* g = g^1\frac{\partial}{\partial \tilde{z}} + g^2\frac{\partial}{\partial \tilde{y}} + g^3\frac{\partial}{\partial \tilde{w}}$, for some smooth functions g^i, satisfying $g^3(0) \neq 0$. Replacing $\tilde{g}$ by $\frac{1}{g^3}\tilde{g}$, we may assume that $\tilde{g} = g^1\frac{\partial}{\partial \tilde{z}} + g^2\frac{\partial}{\partial \tilde{y}} + \frac{\partial}{\partial \tilde{w}}$. Then, since $\tilde{v}_1$ and $\tilde{v}_2$ are symmetries of $\mathcal{D}^0 = \operatorname{span}\left\{g^1\frac{\partial}{\partial \tilde{z}} + g^2\frac{\partial}{\partial \tilde{y}} + \frac{\partial}{\partial \tilde{w}}\right\}$, we have $[\tilde{v}_i, \tilde{g}] \in \mathcal{D}^0$, for $i = 1, 2$, which implies $g^1 = g^1(\tilde{w})$ and $g^2 = g^2(\tilde{w})$. Therefore, we in fact have $[\tilde{v}_1, \tilde{g}] = [\tilde{v}_2, \tilde{g}] = 0$ and thus there exists a local diffeomorphism $(z, y, w) = \phi(\tilde{z}, \tilde{y}, \tilde{w})$ such that $\phi_*\tilde{v}_1 = \frac{\partial}{\partial z}$, $\phi_*\tilde{v}_2 = \frac{\partial}{\partial y}$ and

$\phi_* \tilde{g} = \frac{\partial}{\partial w}$. Denote $v_3 = v_3^1 \frac{\partial}{\partial z} + v_3^2 \frac{\partial}{\partial y} + v_3^3 \frac{\partial}{\partial w}$, the third infinitesimal symmetry, where $v_3^1 = v_3^1(z, y)$ and $v_3^2 = v_3^2(z, y)$ since v_3 is a symmetry of $\mathcal{D}^0 = \operatorname{span}\left\{\frac{\partial}{\partial w}\right\}$.

Using commutativity relations $[v_1, v_3] = 2v_1$ and $[v_2, v_3] = v_2$ of (6.1), which has not been changed by applying diffeomorphisms, we obtain

$$v_3 = (2z + c)\frac{\partial}{\partial z} + (y + d)\frac{\partial}{\partial y} + v_3(w)\frac{\partial}{\partial w}, \quad c, d \in \mathbb{R}.$$

To avoid unnecessary computations, we take $v_3 \leftarrow v_3 - cv_1 - dv_2 \in \mathfrak{L}$, so we can assume that $c = d = 0$. Since v_1 and v_2 are symmetries of $f = f^0 \frac{\partial}{\partial z} + f^1 \frac{\partial}{\partial y}$, we deduce that $f^0 = f^0(w)$ and $f^1 = f^1(w)$. Moreover, using the symmetry v_3 we deduce the following system of equations for the functions $f^0(w)$, $f^1(w)$, and $v_3^3(w)$:

$$(Sys) \; : \left\{ \begin{array}{ll} v_3^3\,(f^0)' - 2f^0 & = 0 \\ v_3^3\,(f^1)' - f^1 & = 0 \end{array} \right. .$$

We will now distinguish two cases.

(a) Assume that $f(0) \notin \mathcal{D}^0(0)$, that is $(f^0(0), f^1(0)) \neq (0, 0)$ and thus by (Sys) we have $((f^0)'(0), (f^1)'(0)) \neq (0, 0)$. Assume $f^1(0) \neq 0$, thus $f^1(w) = c + h(w)$, where $c = f^1(0)$ and $h(0) = 0$. Replacing y by $\frac{y}{c}$ we may assume that $f^1(w) = 1 + h(w)$, where $h'(0) \neq 0$, if not the second equation of (Sys) is not satisfied at $0 \in \mathbb{R}$. Set $\hat{w} = h(w)$ and denote the transformed vector fields $\hat{f}$ and $\hat{v}_3$, for which (Sys) implies $\hat{v}_3^3 = 1 + \hat{w}$ and

$$\left(\hat{f}^0\right)'(1 + \hat{w}) = 2\hat{f}^0.$$

Solving this equation gives $\hat{f}^0(\hat{w}) = c(1 + \hat{w})^2$ with $c \in \mathbb{R}$. But c can not be 0, otherwise the Lie algebra $\mathfrak{L}$ of infinitesimal symmetries would be of infinite dimension, thus not isomorphic to $\mathfrak{L}^0_{1,0}$. Finally, introducing $\hat{z} = \frac{z}{c}$ we obtain $\Sigma^{0,1}_{1,0}$. If $f^1(0) = 0$, then $f^0(0) \neq 0$ implying that $(f^0)'(0) \neq 0$ and leading to the normalisation $\hat{f}^0(\hat{w}) = 1 + \hat{w}$ giving $\hat{v}_3^3 = 2(1 + \hat{w})$ and $\hat{f}^1 = (1 + \hat{w})^{1/2}$. This forms is equivalent to $\Sigma^{0,1}_{1,0}$ by the local diffeomorphism $w = (1+\hat{w})^{1/2} - 1$, sending 0 into 0.

(b) Assume $f(0) \in \mathcal{D}^0(0)$ and $g \wedge \mathrm{ad}_g f(0) \neq 0$. If $(f^1)'(0) \neq 0$, take $(\hat{z}, \hat{y}, \hat{w}) = (z, y, f^1(w))$ as a local diffeomorphism around $0 \in \mathbb{R}^3$ that maps f^0, f^1 and v_3^3 into $\hat{f}^0$, $\hat{f}^1$ and $\hat{v}_3^3$ respectively. We have $\hat{f}^1 = \hat{w}$, so the system (Sys) implies $\hat{v}_3^3 = \hat{w}$ and

$$\hat{w}\left(\hat{f}^0\right)' = 2\hat{f}^0.$$

Solving this equation gives $\hat{f}^0(\hat{w}) = c(\hat{w})^2$ with $c \in \mathbb{R}$. But c can not be 0, otherwise the Lie algebra $\mathfrak{L}$ of infinitesimal symmetries would be of infinite dimension. Finally, introducing $\hat{z} = \frac{z}{c}$ we obtain $\Sigma^{0,0}_{1,0}$. If $(f^1)'(0) = 0$, then $(f^0)'(0) \neq 0$ and by applying the local diffeomorphism $(\hat{z}, \hat{y}, \hat{w}) = (z, y, f^0(w))$ we get $\hat{f}^0 = \hat{w}$ yielding $\hat{v}_3^3(\hat{w}) = 2\hat{w}$ and $2\hat{w}(\hat{f}^1)' = \hat{f}^1$. Hence, $|\hat{f}^1| = d|\hat{w}|^{1/2}$ and the only smooth solution, around $\hat{w} = 0$, is given by $d = 0$ but then the Lie algebra $\mathfrak{L}$ of infinitesimal symmetries would be of infinite dimension contradicting our assumption.

$\blacksquare$

The symmetry algebra of Σ being isomorphic to $\mathfrak{L}^0_{1,0}$ does not completely characterise systems feedback equivalent to $\Sigma^0_{1,0}$. Indeed, there is a small class of systems that are not feedback equivalent to $\Sigma^0_{1,0}$ although their symmetry algebra is isomorphic to $\mathfrak{L}^0_{1,0}$. The following proposition gives such an example by relaxing the assumption $\dim \mathcal{D}^1(\xi_0) = 2$ in statement *(ii)* of the previous theorem.

Proposition 6.2. *Let $\Sigma : \dot{\xi} = f(\xi) + g(\xi)u$ be a control-affine system on a 3-dimensional smooth manifold with a scalar control, and let $\mathfrak{L}$ be its Lie algebra of infinitesimal symmetries. Assume $f(\xi_0) \in \mathcal{D}^0(\xi_0)$ and, additionally, that there exists $k \geq 1$, the smallest integer such that $g \wedge \mathrm{ad}^k_g f(\xi_0) \neq 0$. Then $\mathfrak{L} \cong \mathfrak{L}^0_{1,0}$ and $\mathfrak{I}(\xi_0) \oplus \mathcal{D}^0(\xi_0) = T_{\xi_0}\mathbb{R}^3$ if and only if Σ is feedback equivalent to*

$$\Sigma^{0,0,k}_{1,0} : \begin{cases} \dot{z} &= w^{2k} \\ \dot{y} &= w^k \\ \dot{w} &= u \end{cases}$$

around $(z_0, y_0, 0)$.

Moreover, it is a classical fact that (under the above assumptions) the integer k is an invariant of feedback, hence if $k \neq k'$, then $\Sigma^{0,0,k}_{1,0}$ and $\Sigma^{0,0,k'}_{1,0}$ are not locally feedback equivalent around $w_0 = 0$. Notice that if $k \geq 2$, then $\Sigma^{0,0,k}_{1,0}$ is a prolongation of the control-nonlinear system $\begin{cases} \dot{z} = w^{2k} \\ \dot{y} = w^k \end{cases}$ and the latter is not a regular parametrisation of the parabolic submanifold $\mathcal{S}^0_{1,0} = \{\dot{z} - \dot{y}^2 = 0\}$ because $\frac{\partial F}{\partial w}(z_0, y_0, 0) = 0$, where $F(x, w) = w^{2k}\frac{\partial}{\partial z} + w^k \frac{\partial}{\partial y}$. To be consistent with the notation of the above proposition, the previously considered normal form $\Sigma^{0,0}_{1,0}$ should actually be denoted $\Sigma^{0,0,1}_{1,0}$.

Proof. We prove the necessity only, as there are no difficulties to show that the Lie algebra of infinitesimal symmetries of $\Sigma^{0,0,k}_{1,0}$ is isomorphic to $\mathfrak{L}^0_{1,0}$. We adapt the point (b) of the proof of Theorem 6.1. Assume that $f(0) \in \mathcal{D}^0(0)$ and that $k \geq 1$ is the smallest integer such that $((f^0)^{(k)}(0), (f^1)^{(k)}(0)) \neq (0, 0)$. If $(f^1)^{(k)}(0) \neq 0$, we can suppose $f^1(0) > 0$ (if not, replace y by $-y$), apply the local diffeomorphism, around $0 \in \mathbb{R}^3$, $(\hat{z}, \hat{y}, \hat{w}) = (z, y, f^1(w)^{1/k})$ that maps f^0, f^1 and v^3_3 into $\hat{f}^0$, $\hat{f}^1$ and $\hat{v}^3_3$, respectively. We have $\hat{f}^1 = \hat{w}^k$, so the system (Sys) implies $\hat{v}^3_3 = \frac{\hat{w}}{k}$ and

$$\hat{w}\left(\hat{f}^0\right)' = 2k\hat{f}^0.$$

Solving this equation gives $\hat{f}^0(\hat{w}) = c\hat{w}^{2k}$ with $c \in \mathbb{R}$. However, the solution passing through $\hat{w} = 0$ is not unique so à priori we may have different values of c for $\hat{w} < 0$ and $\hat{w} > 0$ but the only C^∞ solutions are those given by the same value of c (either $c = 0$ or $c \neq 0$) for any $\hat{w}$. But c can not be 0, otherwise the Lie algebra $\mathfrak{L}$ of infinitesimal symmetries would be of infinite dimension. Finally, introducing $\hat{z} = \frac{z}{c}$ we obtain the desired form $\Sigma^{0,0,k}_{1,0}$.

If $(f^0)^{(k)}(0) \neq 0$, then normalising $f^0 = \hat{w}^k$ and applying an analogous procedure we deduce that $f^1(\hat{w}) = c|\hat{w}|^{k/2}$. If $c = 0$ then the Lie algebra of infinitesimal symmetries would be of infinite dimension contradicting our assumptions. In all other cases of k and c the solution is not smooth around $\hat{w} = 0$ except for $k = 2l$ and the same value of c for $\hat{w} < 0$ and $\hat{w} > 0$. But in the latter case we have $(f^1)^{(l)}(0) \neq 0$, with $l < k$, contradicting the definition of k. ■

Remark. Theorem 6.1 statement *(ii)* and Proposition 6.2 describe all systems having $\mathfrak{L}^0_{1,0}$ as the symmetry algebra for which k exists (in particular, all analytic systems). In the C^∞ category there are, however, systems for which k does not exist and the symmetry algebra is $\mathfrak{L}^0_{1,0}$. For example, consider

$$\begin{cases} \dot{z} &= \mathfrak{f}(w)^2 \\ \dot{y} &= \mathfrak{f}(w) \\ \dot{w} &= u \end{cases}, \quad \text{with} \quad \mathfrak{f}(w) = \exp\left(-\frac{1}{w^2}\right), \, \mathfrak{f}(0) = 0.$$

By a straightforward calculation, its symmetry algebra is, indeed, $\mathfrak{L}^0_{1,0}$ but, obviously, k does not exist at $(z_0, y_0, 0)$. ◆

2 Characterisation of null-forms (p, q)-paraboloid systems by their infinitesimal symmetries

We now extend the theorem of the previous section for general null-form (p, q)-paraboloid systems given by

$$\Sigma^0_{p,q} \; : \; \begin{cases} \dot{z} &= \sum_{i=1}^m \varepsilon_i (w_i)^2 \\ \dot{y}_i &= w_i \\ \dot{w}_i &= u_i \end{cases},$$

where $m = p + q$ and $\varepsilon_i = \begin{cases} +1 & 1 \leq i \leq p \\ -1 & p+1 \leq i \leq m \end{cases}$. Without loss of generality we assume that $p \geq q$. When $m = 1$, then $\Sigma^0_{1,0}$ is the form studied in the previous section; when $m = 2$, we either have $\Sigma^0_{2,0}$ a p-elliptic system, or $\Sigma^0_{1,1}$ a p-hyperbolic system, which were characterised and studied in Chapter 4. The Lie algebra of infinitesimal symmetries of $\Sigma^0_{p,q}$ is given by (see a detailed computation in Appendix 6.A):

$$\text{(6.2)} \quad \mathfrak{L}^0_{p,q} = \text{vect}_{\mathbb{R}} \left\{ \frac{\partial}{\partial z}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_m} \right\} \oplus \text{vect}_{\mathbb{R}} \left\{ 2z\frac{\partial}{\partial z} + \sum_{i=1}^m \left(y_i \frac{\partial}{\partial y_i} + w_i \frac{\partial}{\partial w_i} \right) \right\}$$
$$\oplus \, \text{vect}_{\mathbb{R}} \left\{ \varepsilon_k y_l \frac{\partial}{\partial y_k} - \varepsilon_l y_k \frac{\partial}{\partial y_l} + \varepsilon_k w_l \frac{\partial}{\partial w_k} - \varepsilon_l w_k \frac{\partial}{\partial w_l}, \; 1 \leq k < l \leq m \right\}.$$

The dimension of this Lie algebra is $\frac{1}{2}(m^2 + m + 4)$, observe that $\dim \mathfrak{L}^0_{p,q} = 2m+1$ (dimension of the state space) if and only if $m = 1$ or $m = 2$. The elements of the above basis of $\mathfrak{L}^0_{p,q}$ will be denoted as follows:

$$v_0 = \frac{\partial}{\partial z}, \quad v_i = \frac{\partial}{\partial y_i} \quad \text{for} \quad 1 \leq i \leq m, \quad E = 2z\frac{\partial}{\partial z} + \sum_{i=1}^m \left(y_i \frac{\partial}{\partial y_i} + w_i \frac{\partial}{\partial w_i} \right),$$
$$\text{and} \quad \Delta_{kl} = \varepsilon_k y_l \frac{\partial}{\partial y_k} - \varepsilon_l y_k \frac{\partial}{\partial y_l} + \varepsilon_k w_l \frac{\partial}{\partial w_k} - \varepsilon_l w_k \frac{\partial}{\partial w_l}, \quad \text{for} \quad 1 \leq k < l \leq m.$$

Moreover we will adopt the following convention $\Delta_{lk} = -\Delta_{kl}$ so that Δ_{kl} can be defined for any pair of indices (k,l). The interpretation of each element of this basis is the following. The vector fields v_0 and v_i being symmetries of $\Sigma^0_{p,q} = (f,g)$ implies that the drift f is invariant under translations of the form $z \mapsto z+a$ and $y_i \mapsto y_i+b_i$ (with $a, b_i \in \mathbb{R}$), which is obvious since $f = f(w)$; E behaves like an Euler vector field, i.e. it encodes the homogeneity degree (with respect to the variable w) of the components of f, meaning that the homogeneity degree of the component along $\frac{\partial}{\partial z}$ is twice the degree of the components along $\frac{\partial}{\partial y_i}$. Finally, the expression of vector fields Δ_{kl} implies that the system is invariant under infinitesimal rotations (either trigonometric or hyperbolic depending on the signs of the ε_k's), which is a less trivial fact but not surprising. Notice that when $m = 1$ then the vector fields Δ_{kl}'s do not exist and when $m = 2$ there is only one of them, namely Δ_{12}. The Lie algebra $\mathfrak{L}^0_{p,q}$ admits the following multiplication table (see Appendix 6.B for details of the computation):

$$\begin{array}{llll} [v_0, v_i] = 0, & [v_0, E] = 2v_0, & [v_0, \Delta_{kl}] = 0, & [E, \Delta_{kl}] = 0, \\ [v_i, v_j] = 0, & [v_i, E] = v_i, & [v_i, \Delta_{kl}] = \varepsilon_k \delta^l_i v_k - \varepsilon_l \delta^k_i v_l, & \end{array} \tag{6.3}$$

where δ^j_i is the Kronecker symbol, and the only non-zero relations between the Δ_{kl} are given by

$$\forall\, 1 \leq i < j < l \leq m, \quad [\Delta_{ij}, \Delta_{il}] = -\varepsilon_i \Delta_{jl}. \tag{6.4}$$

Therefore, $\mathfrak{L}^0_{p,q}$ is composed of two abelian ideals, $\mathfrak{I}_1 = \mathrm{vect}_{\mathbb{R}}\{v_0\}$ and $\mathfrak{I}_m = \mathrm{vect}_{\mathbb{R}}\{v_1, \ldots, v_m\}$, and of a subalgebra $\mathfrak{l} = \mathfrak{E} \oplus \Delta$, where $\mathfrak{E} = \mathrm{vect}_{\mathbb{R}}\{E\}$ and $\Delta = \mathrm{vect}_{\mathbb{R}}\{\Delta_{kl}\}$. Moreover, denoting the sum $\mathfrak{I} = \mathfrak{I}_1 \oplus \mathfrak{I}_m$, we see that $\mathfrak{E} \oplus \mathfrak{I}$ is also an ideal of $\mathfrak{L}$ (so $\mathfrak{E}$ is an ideal of the subalgebra $\mathfrak{l}$). Using the computation at the end of Appendix 6.C, we conclude that Δ is isomorphic to the Lie algebra $\mathfrak{so}(p,q)$ (of the indefinite orthogonal group $O(p,q)$). We tried to give a characterisation of the Lie algebra $\mathfrak{L}^0_{p,q}$ among all Lie algebras of the same dimension (as we did for $\mathfrak{L}^0_{1,0}$ in Proposition 6.1), we have a result for the case $m = 2$ (that case is particular because the Lie algebras $\mathfrak{L}^0_{2,0}$ and $\mathfrak{L}^0_{1,1}$ are solvable) but we have not yet succeeded in proposing a characterisation of the general Lie algebra $\mathfrak{L}^0_{p,q}$.

We now state our main theorem establishing that $\mathfrak{L}^0_{p,q}$ determines the null-form (p,q)-paraboloids systems $\Sigma^0_{p,q}$. Recall that we attach to a control-affine system $\Sigma = (f,g)$ the following distributions:

$$\mathcal{D}^0 = \mathrm{span}\{g_1, \ldots, g_m\} \quad \text{and} \quad \mathcal{D}^1 = \mathcal{D}^0 + \left[f, \mathcal{D}^0\right].$$

Theorem 6.2 ($\mathfrak{L}^0_{p,q}$ characterises $\Sigma^0_{p,q}$). *Let $\Sigma : \dot{\xi} = f(\xi) + g(\xi)u$, locally around ξ_0, be a control-affine system on a $(2m+1)$-dimensional manifold with control $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$, and let $\mathfrak{L}$ be its Lie algebra of infinitesimal symmetries.*

(i) Σ is feedback equivalent to $\Sigma^0_{p,q}$ around $(z_0, y_0, 0)$ if and only if

$$\mathfrak{L} \cong \mathfrak{L}^0_{p,q}, \quad \mathcal{D}^0 \text{ is involutive}, \quad \mathfrak{I}(\xi_0) \oplus \mathcal{D}^0(\xi_0) = T_{\xi_0}\mathbb{R}^{2m+1},$$
$$\dim \mathcal{D}^1(\xi_0) = 2m, \quad \text{and} \quad f(\xi_0) \in \mathcal{D}^0(\xi_0).$$

(ii) Σ is feedback equivalent to $\Sigma^0_{p,q}$ around (z_0, y_0, w_0), with $w_0 \neq 0$, if and only if

$$\mathfrak{L} \cong \mathfrak{L}^0_{p,q}, \quad \mathcal{D}^0 \text{ is involutive}, \quad \mathcal{D}^0(\xi_0) \oplus \mathfrak{I}(\xi_0) = T_{\xi_0}\mathbb{R}^{2m+1},$$
$$\dim \mathcal{D}^1(\xi_0) = 2m, \quad \text{and} \quad f(\xi_0) \notin \mathcal{D}^0(\xi_0).$$

Remark (Comparison with the case $m = 1$). Observe that the above theorem applies for any $m \geq 1$, hence it is a generalisation of Theorem 6.1 of the previous section. However, when $m = 1$, the assumptions can be simplified. Indeed, in that case we automatically get that $\mathcal{D}^0 = \operatorname{span}\{g_1\}$ is involutive. Moreover, for statement *(i)* we showed that $\dim \mathcal{D}^1(0) = 2$ is a consequence of the other assumptions. ◆

Remark (Difference between the two cases). The theorem indicates that the points $w_0 \neq 0$ and $w_0 = 0$ are distinguished for $\Sigma^0_{p,q}$. Clearly, $w_0 = 0$ is an equilibrium point for $\Sigma^0_{p,q}$ whereas $w_0 \neq 0$ is not. As a consequence, $\Sigma^0_{p,q}$ admits several non-equivalent local normal forms, around $w_0 = 0$, given by

$$\Sigma^{0,\eta}_{p,q} : \begin{cases} \dot{z} &= \sum_{i=1}^m \varepsilon_i (w_i + \eta_i)^2 \\ \dot{y}_i &= w_i + \eta_i \\ \dot{w}_i &= u_i \end{cases},$$

where $\eta_i = 0$ for all $i = 1, \ldots, m$ in statement *(i)* and for the second statement, there is at least one index $1 \leq i \leq m$ such that $\eta_i \neq 0$. ◆

Proof. We only show the sufficiency part of the theorem as the necessity follows immediately from the description of the Lie algebra of infinitesimal symmetries given above and from the fact that all conditions are invariants under feedback transformations.

The strategy of the proof is the following. We will deal with statements *(i)* and *(ii)* together. First, we will deduce as much facts as possible on the basis of a Lie algebra of symmetries $\mathfrak{L}$ of $\Sigma = (f, g)$ and on the vector fields g_i. Second, using all of those properties we will deduce conditions on the components of the vector field f. Finally, we will show that there exists a coordinate system in which Σ takes the form of $\Sigma^{0,\eta}_{p,q}$.

Consider a system $\Sigma = (f, g)$ given by vector fields f and $g = (g_1, \ldots, g_m)$ and let vector fields $\{v_0, v_i, E, \Delta_{kl}\}$ span its $\frac{1}{2}(m^2 + m + 4)$-dimensional Lie algebra $\mathfrak{L}$ of infinitesimal symmetries which by assumption is isomorphic to $\mathfrak{L}^0_{p,q}$. We can assume that this basis satisfies the commutativity relations of $\mathfrak{L}^0_{p,q}$ given by (6.3) and (6.4) and, in particular, the abelian ideal of $\mathfrak{L}$ is $\mathfrak{I} = \operatorname{vect}_{\mathbb{R}}\{v_0, v_1, \ldots, v_m\}$.

Since $\mathfrak{I}(\xi_0) \oplus \mathcal{D}^0(\xi_0) = T_{\xi_0}\mathbb{R}^{2m+1}$, it follows that $v_0, v_1, \ldots, v_m$ and $g_1, \ldots, g_m$ are independent locally around ξ_0. We apply a local diffeomorphism $\psi(\xi) = (\tilde{z}, \tilde{y}, \tilde{w})$ such that $\psi(\xi_0) = 0 \in \mathbb{R}^{2m+1}$, and that

$$\tilde{v}_0 = \psi_* v_0 = \frac{\partial}{\partial \tilde{z}}, \quad \tilde{v}_i = \psi_* v_i = \frac{\partial}{\partial \tilde{y}_i} \quad \text{for} \quad 1 \leq i \leq m,$$
$$\text{and} \quad \tilde{g}_i = \psi_* g_i = \tilde{g}^0_i \frac{\partial}{\partial \tilde{z}} + \sum_{j=1}^m \tilde{g}^j_i \frac{\partial}{\partial \tilde{y}_j} + \tilde{\mathrm{G}}^j_i \frac{\partial}{\partial \tilde{w}_j} \quad \text{for} \quad 1 \leq i \leq m.$$

Since $\tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_m, \tilde{g}_1, \ldots, \tilde{g}_m$ are locally independent around $0 \in \mathbb{R}^{2m+1}$ it follows that $\det \tilde{\mathrm{G}} \neq 0$, where $\tilde{\mathrm{G}}$ is the matrix formed by the functions $\tilde{\mathrm{G}}^j_i$.

Thus, using the feedback $\tilde{u} = \tilde{\mathrm{G}}u$, we may assume that $\tilde{g}_i = \tilde{g}^0_i \frac{\partial}{\partial \tilde{z}} + \tilde{g}^j_i \frac{\partial}{\partial \tilde{y}_j} + \frac{\partial}{\partial \tilde{w}_i}$. As the vector fields $\tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_m$ are symmetries of $\mathcal{D}^0$, we conclude from $[\tilde{v}_i, \tilde{g}_j] \in \mathcal{D}^0$ that $\tilde{g}^0_i = \tilde{g}^0_i(\tilde{w})$ and that $\tilde{g}^j_i = \tilde{g}^j_i(\tilde{w})$. Therefore we actually have $[\tilde{v}_i, \tilde{g}_j] = 0$ for all i, j. Moreover, since $\mathcal{D}^0$ is involutive, we have $[\tilde{g}_i, \tilde{g}_j] \in \mathcal{D}^0$ for all i, j, and by a straightforward computation we deduce that it actually implies $[\tilde{g}_i, \tilde{g}_j] = 0$. Thus, locally around $0 \in \mathbb{R}^{2m+1}$, there exists a diffeomorphism $(z, y, w) = \phi(\tilde{z}, \tilde{y}, \tilde{w})$ such that

$$v_0 = \phi_* \tilde{v}_0 = \frac{\partial}{\partial z}, \quad v_i = \phi_* \tilde{v}_i = \frac{\partial}{\partial y_i}, \quad \text{and} \quad g_i = \phi_* \tilde{g}_i = \frac{\partial}{\partial w_i}. \tag{6.5}$$

Using the commutativity relations of $\mathfrak{L}$ given by (6.3), which has not been changed by applying diffeomorphisms, and the fact that E is a symmetry of $\mathcal{D}^0$ (meaning that $[E, g_i] = \left[E, \frac{\partial}{\partial w_i}\right] \in \mathcal{D}^0$ holds) we obtain

$$E = (2z + a_0)\frac{\partial}{\partial z} + \sum_{i=1}^{m}(y_i + a_i)\frac{\partial}{\partial y_i} + \sum_{i=1}^{m} e_i(w)\frac{\partial}{\partial w_i}, \quad a_0, a_i \in \mathbb{R}.$$

To simplify computations we replace E by $E - a_0 v_0 - \sum_{i=1}^{m} a_i v_i$, that remains to be an element of the symmetry algebra $\mathfrak{L}$, so we can remove the constants a_0 and a_i without changing commutativity relations (6.3). Denoting

$$\Delta_{kl} = \delta^0_{kl}\frac{\partial}{\partial z} + \sum_{j=1}^{m}\left(\delta^j_{kl}\frac{\partial}{\partial y_j} + d^j_{kl}\frac{\partial}{\partial w_j}\right)$$

we deduce, using multiplication table (6.3), the following facts about functions δ^0_{kl}, δ^j_{kl}, and d^j_{kl}:

$$[v_0, \Delta_{kl}] = 0 \quad \Rightarrow \quad \delta^0_{kl} = \delta^0_{kl}(y, w), \quad \delta^j_{kl} = \delta^j_{kl}(y, w), \quad d^j_{kl} = d^j_{kl}(y, w),$$

$$[v_i, \Delta_{kl}] = \varepsilon_k \delta^l_i v_k - \varepsilon_l \delta^k_i v_l \quad \Rightarrow \quad \delta^0_{kl} = \delta^0_{kl}(w), \quad d^j_{ij} = d^j_{kl}(w),$$

$$\text{and} \quad \delta^j_{kl}(y, w) = \begin{cases} \bar{\delta}^j_{kl}(w) & \text{if} \quad j \neq k, \text{ and } j \neq l \\ \varepsilon_k y_l + \bar{\delta}^k_{kl}(w) & \text{if} \quad j = k \\ -\varepsilon_l y_k + \bar{\delta}^l_{kl}(w) & \text{if} \quad j = l \end{cases}, \tag{6.6}$$

$$[E, \Delta_{kl}] = 0 \quad \Rightarrow \quad \delta^0_{kl} = 0.$$

Consider the vector field $f = f^0\frac{\partial}{\partial z} + \sum_{j=1}^{m} f^j \frac{\partial}{\partial y_j}$, where the components along $\frac{\partial}{\partial w}$ have been removed by a feedback of the form $f \mapsto f + g\alpha$, then by the fact that $v_0 = \frac{\partial}{\partial z}$ and $v_i = \frac{\partial}{\partial y_i}$ are symmetries of Σ we get that $f^0 = f^0(w)$ and $f^j = f^j(w)$. Moreover, using E as a symmetry of f, i.e. $[E, f] = 0 \mod \mathcal{D}^0$, we get the following first order partial differential equations around $w = 0$:

$$\sum_{i=1}^{m} e_i(w)\frac{\partial f^0}{\partial w_i} = 2f^0, \quad \text{and} \quad \sum_{i=1}^{m} e_i(w)\frac{\partial f^j}{\partial w_i} = f^j \quad \text{for} \quad 1 \leq j \leq m. \tag{6.7}$$

Using Δ_{kl} as a symmetry of f, i.e. $[\Delta_{kl}, f] = 0 \mod \mathcal{D}^0$, we get

$$\left[\sum_{i=1}^{m} \delta^i_{kl}(y_k, y_l, w)\frac{\partial}{\partial y_i} + d^i_{kl}(w)\frac{\partial}{\partial w_i}, f^0(w)\frac{\partial}{\partial z} + \sum_{j=1}^{m} f^j(w)\frac{\partial}{\partial y_j}\right] = 0 \mod \mathcal{D}^0,$$

$$\Rightarrow \sum_{i=1}^{m} d^i_{kl}\frac{\partial f^0}{\partial w_i}\frac{\partial}{\partial z} + \sum_{i=1}^{m}\sum_{j=1}^{m} -f^j\frac{\partial \delta^i_{kl}}{\partial y_j}\frac{\partial}{\partial y_i} + d^i_{kl}\frac{\partial f^j}{\partial w_i}\frac{\partial}{\partial y_j} = 0,$$

$$\Rightarrow \sum_{i=1}^{m} d^i_{kl}\frac{\partial f^0}{\partial w_i} = 0, \quad \text{and} \quad \sum_{i=1}^{m}\sum_{j=1}^{m} d^i_{kl}\frac{\partial f^j}{\partial w_i}\frac{\partial}{\partial y_j} - f^j\frac{\partial \delta^i_{kl}}{\partial y_j}\frac{\partial}{\partial y_i} = 0,$$

$$\Rightarrow \sum_{j=1}^{m}\sum_{i=1}^{m} d^i_{kl}\frac{\partial f^j}{\partial w_i}\frac{\partial}{\partial y_j} = \varepsilon_k f^l\frac{\partial}{\partial y_k} - \varepsilon_l f^k\frac{\partial}{\partial y_l}.$$

Thus $f = f^0(w)\frac{\partial}{\partial z} + \sum_{j=1}^m f^j(w)\frac{\partial}{\partial y_j}$ together with the functions $e_i(w)$ of E and the functions d^i_{kl} of Δ_{kl} satisfy (6.7) and for all $1 \le k < l \le m$:

$$\sum_{i=1}^m d^i_{kl}\frac{\partial f^0}{\partial w_i} = 0, \quad \text{and} \quad \begin{cases} \sum_{i=1}^m d^i_{kl}\frac{\partial f^j}{\partial w_i} & = 0 & \text{if} \quad j \neq k,\ j \neq l, \\ \sum_{i=1}^m d^i_{kl}\frac{\partial f^k}{\partial w_i} & = \varepsilon_k f^l & \text{if} \quad j = k, \\ \sum_{i=1}^m d^i_{kl}\frac{\partial f^l}{\partial w_i} & = -\varepsilon_l f^k & \text{if} \quad j = l. \end{cases} \tag{6.8}$$

We now show that we can construct a local diffeomorphism around $0 \in \mathbb{R}^{2m+1}$ such that Σ takes the form of $\Sigma^{0,\eta}_{p,q}$ in those coordinates. We denote $\eta := f(0) \in \mathbb{R}^{m+1}$, i.e. $\eta_j = f^j(0)$ with $0 \le j \le m$. By assumption $\dim \mathcal{D}^1(0) = 2m$ thus rk $\left(\frac{\partial f}{\partial w}\right) = m$. We will show at the end of the proof that our conditions always imply that we have

$$\det \begin{pmatrix} \frac{\partial f^1}{\partial w_1} & \cdots & \frac{\partial f^1}{\partial w_m} \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial w_1} & \cdots & \frac{\partial f^m}{\partial w_m} \end{pmatrix} \neq 0, \tag{6.9}$$

so we can take $(\hat{z}, \hat{y}, \hat{w}_1, \ldots, \hat{w}_m) = (z, y, f^1(w) - \eta_1, \ldots, f^m(w) - \eta_m)$ as a local diffeomorphism around $0 \in \mathbb{R}^{2m+1}$. In order to simplify the notations, the coordinates $(\hat{z}, \hat{y}, \hat{w})$ are simply denoted (z, y, w). In this coordinate system, we have $f^j(w) = w_j + \eta_j$, for $1 \le j \le m$, and the conditions of (6.7) imply that $e_j(w) = w_j + \eta_j$, for $1 \le j \le m$, and

$$2f^0(w) = \sum_{i=1}^m (w_i + \eta_i)\frac{\partial f^0}{\partial w_i}. \tag{6.7'}$$

The right hand side of this equation is seen as the Lie derivative of $f^0(w)$ along the Euler vector field $\sum_{i=1}^m (w_i + \eta_i)\frac{\partial}{\partial w_i}$ and thus, by Euler's homogeneous function theorem, it implies that f^0 is homogeneous of degree 2 with respect to the coordinates $(w_i + \eta_i)$ (see [GS64] for an introduction to homogeneous functions). Inserting $f^j(w) = w_j + \eta_j$, for $1 \le j \le m$, in the second condition of (6.8), implies

$$d^i_{kl} = \begin{cases} 0 & \text{if} \quad i \neq k \text{ and } i \neq l \\ \varepsilon_k(w_l + \eta_l) & \text{if} \quad i = k \\ -\varepsilon_l(w_k + \eta_k) & \text{if} \quad i = l \end{cases}.$$

Therefore, using the first equation of (6.8), we conclude that f^0 is a homogeneous function of degree 2 satisfying

$$\varepsilon_k(w_l + \eta_l)\frac{\partial f^0}{\partial w_k} - \varepsilon_l(w_k + \eta_k)\frac{\partial f^0}{\partial w_l} = 0, \quad \text{for all} \quad 1 \le k < l \le m. \tag{6.10}$$

We will show that $f^0 = F^0\left(\sum_{i=1}^m \varepsilon_i(w_i + \eta_i)^2\right)$ for some smooth function F^0. To this end, we solve (6.10) for $k = 1$ and for $2 \le l \le m$ iteratively:

$$\varepsilon_1(w_2 + \eta_2)\frac{\partial f^0}{\partial w_1} - \varepsilon_2(w_1 + \eta_1)\frac{\partial f^0}{\partial w_2} = 0 \quad \Rightarrow \quad f^0 = F^0_{12}(W_{12}, w_3, \ldots, w_m),$$

where $W_{12} = \varepsilon_1(w_1 + \eta_1)^2 + \varepsilon_2(w_2 + \eta_2)^2$. Next,

$$\varepsilon_1(w_3 + \eta_3)\frac{\partial f^0}{\partial w_1} - \varepsilon_3(w_1 + \eta_1)\frac{\partial f^0}{\partial w_3} = 0 \quad \Rightarrow \quad 2(w_3 + \eta_3)\frac{\partial F^0_{12}}{\partial W_{12}} - \varepsilon_3\frac{\partial F^0_{12}}{\partial w_3} = 0,$$
$$\Rightarrow \quad F^0_{12} = F^0_{123}(W_{123}, w_4, \ldots, w_m).$$

with $W_{123} = \varepsilon_1(w_1+\eta_1)^2 + \varepsilon_2(w_2+\eta_2)^2 + \varepsilon_3(w_3+\eta_3)^2$. And so on. Thus we obtain $f^0 = F^0(W)$, where $F^0 = F^0_{1\cdots m}$ is an arbitrary smooth function of the variable $W = W_{1\cdots m} = \sum_{i=1}^m \varepsilon_i(w_i+\eta_i)^2$. By inserting f^0 of that form into (6.7'), we conclude that

$$2f^0 = \sum_{i=1}^m (w_i+\eta_i)\frac{\partial f^0}{\partial w_i} \quad \Rightarrow \quad F^0 = W\frac{\mathrm{d}F^0}{\mathrm{d}W} \quad \Rightarrow \quad F^0(W) = \lambda \cdot W,$$

where $\lambda \in \mathbb{R}$. We necessarily have $\lambda \neq 0$, otherwise we would have $f^0 \equiv 0$ and the Lie algebra of symmetries would be of infinite dimension contradicting our assumption. By replacing z by $\frac{z}{\lambda}$, we normalise the z-component and we see that we have transformed Σ into

$$\begin{cases} \dot{z} &= \sum_{i=1}^m \varepsilon_i(w_i+\eta_i)^2 \\ \dot{y}_i &= w_i+\eta_i \\ \dot{w}_i &= u_i \end{cases}, \quad \text{around} \quad 0 \in \mathbb{R}^{2m+1}.$$

Thus for both statements *(i)* and *(ii)* of Theorem 6.2 we showed that Σ is equivalent to $\Sigma^{0,\eta}_{p,q}$.

To terminate the proof, it remains to show that (6.9) either always holds or, if not, that it still leads to an equivalent description of $\Sigma^{0,\eta}_{p,q}$. By contradiction, assume that $\det\begin{pmatrix} \frac{\partial f^1}{\partial w_1} & \cdots & \frac{\partial f^1}{\partial w_m} \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial w_1} & \cdots & \frac{\partial f^m}{\partial w_m}\end{pmatrix} = 0$ at $w=0$. Since we suppose that $\dim \mathcal{D}^1(0) = 2m$, it follows that we necessarily have, around $w=0$,

$$\det\begin{pmatrix} \frac{\partial f^0}{\partial w_1} & \cdots & \frac{\partial f^0}{\partial w_m} \\ \frac{\partial f^2}{\partial w_1} & \cdots & \frac{\partial f^2}{\partial w_m} \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial w_1} & \cdots & \frac{\partial f^m}{\partial w_m}\end{pmatrix} \neq 0, \quad \text{if needed renumber the } y\text{-components of } \Sigma.$$

We, therefore, introduce around $0 \in \mathbb{R}^{2m+1}$ a local coordinate system defined by $(\hat{z},\hat{y},\hat{w}_1,\hat{w}_2,\ldots,\hat{w}_m) = (z,y,f^0(w)-\eta_0, f^2(w)-\eta_2,\ldots,f^m(w)-\eta_m)$. To simplify notations we denote $(\hat{z},\hat{y},\hat{w})$ again by (z,y,w). Then, the conditions of (6.7) imply that $e_1(w) = 2(w_1+\eta_0)$, $e_i(w) = w_i+\eta_i$ for $2 \le i \le m$, and

$$\text{(6.7'')} \qquad 2(w_1+\eta_0)\frac{\partial f^1}{\partial w_1} + \sum_{j=2}^m (w_j+\eta_j)\frac{\partial f^1}{\partial w_j} = f^1.$$

Condition (6.8) yields $d^1_{kl} = 0$, for all $1 \le k < l \le m$, and

$$\text{(6.11)} \qquad \begin{cases} d^j_{kl} = 0 & \text{if} \quad j \neq k,\ j \neq l \\ d^k_{kl} = \varepsilon_k(w_l+\eta_l) & \text{if} \quad j = k \\ d^l_{kl} = -\varepsilon_l(w_k+\eta_k) & \text{if} \quad j = l \end{cases} \qquad \text{for all} \quad 2 \le k < l \le m.$$

We now separate cases by the dimension m.

(i) Suppose $m = 1$. In the case $\eta_0 \neq 0$, we showed in the proof of Theorem 6.1 of the previous section that equation (6.7") can smoothly be solved around $w = 0$. That solution leaded to a system equivalent to $\Sigma^{0,1}_{1,0}$. On the other hand, when $\eta_0 = 0$, we showed that equation (6.7") does not admit nontrivial smooth solutions, which is a contradiction with the finite dimension of $\mathfrak{L}$.

The arguments for the cases $m = 2$ and $m \geq 3$ are slightly different. The difference between the two cases is that in the former, the Lie algebras $\mathfrak{so}(2)$ and $\mathfrak{so}(1,1)$ are one-dimensional Lie algebras. In the case $m = 2$, we will show that the existence of smooth solutions of (6.7") would imply that Σ is linearisable and therefore its Lie algebra of symmetries would be of infinite dimension, a contradiction. And, in the case $m \geq 3$ we will show that the condition $d^1_{kl} = 0$ contradicts the fact that $\Delta \cong \mathfrak{so}(p,q)$.

(ii) Assume that $m = 2$, so we either have $(p,q) = (2,0)$ or $(p,q) = (1,1)$. Our reasoning up here sums up to the following description of the fields of $\mathfrak{L}$ and of Σ:

$$f^0 = w_1 + \eta_0, \quad f^2 = w_2 + \eta_2, \quad e_1 = 2(w_1 + \eta_0), \quad e_2 = w_2 + \eta_2, \quad d^1_{12} = 0,$$
$$d^2_{12} = -\varepsilon_2 f^1, \quad d^2_{12}\frac{\partial f^1}{\partial w_2} = \varepsilon_1(w_2 + \eta_2), \quad 2(w_1 + \eta_0)\frac{\partial f^1}{\partial w_1} + (w_2 + \eta_2)\frac{\partial f^1}{\partial w_2} = f^1.$$

Recall also that we have the assumption $\det \begin{pmatrix} \frac{\partial f^1}{\partial w_1} & \frac{\partial f^1}{\partial w_2} \\ \frac{\partial f^2}{\partial w_1} & \frac{\partial f^2}{\partial w_2} \end{pmatrix} \Bigg|_{w=0} = 0$ implying that $\frac{\partial f^1}{\partial w_1}\Big|_{w=0} = 0$. We show that our assumptions imply that $f^1 = \lambda(w_2 + \eta_0)$ which would then lead to an infinite dimensional Lie algebra of symmetries, contradicting our assumption. Solutions of

$$-\varepsilon_2 f^1 \frac{\partial f^1}{\partial w_2} = \varepsilon_1(w_2 + \eta_2), \quad \text{around } w = 0,$$

are given by $f^1(w) = \sqrt{-\varepsilon_1\varepsilon_2(w_2 + \eta_2)^2 + F_1(w_1)}$. Using equation (6.7") given by

$$2(w_1 + \eta_0)\frac{\partial f^1}{\partial w_1} + (w_2 + \eta_2)\frac{\partial f^1}{\partial w_2} = f^1,$$

we conclude that $F_1(w_1)$ satisfies $(w_1 + \eta_0)\frac{\partial F_1}{\partial w_1} = F_1$ implying that $F_1(w_1) = \lambda(w_1 + \eta_0)$ where $\lambda \in \mathbb{R}$. Since we assume $\frac{\partial f_1}{\partial w_1}(w = 0) = 0$, it implies that $\frac{\partial F_1}{\partial w_1}(w_1 = 0) = 0$, which is possible only if $\lambda = 0$. Therefore, we have shown that we necessarily have $f^1 = f^1(w_2) = \sqrt{-\varepsilon_1\varepsilon_2}(w_2 + \eta_2)$. But then, system Σ would be linear (or even non-existing) and thus its Lie algebra of symmetries would be of infinite dimension, contradicting our assumption.

(iii) Assume that $m \geq 3$. We show that the vector fields (recall that we obtained $d^1_{kl} = 0$, see the observation just above (6.11))

$$\Delta_{kl} = \sum_{j=1}^{m} \delta^j_{kl}\frac{\partial}{\partial y_j} + \sum_{j=2}^{m} d^j_{kl}\frac{\partial}{\partial w_j},$$

with δ^j_{kl} and d^j_{kl} defined as in (6.6) and (6.11), cannot generate the Lie algebra $\mathfrak{so}(p,q)$. Indeed for any $1 < j < l \leq m$ (justifying that we need $m \geq 3$), we have

$$\begin{aligned}
[\Delta_{1j}, \Delta_{1l}] = \Bigg[&(\varepsilon_1 y_j + \bar\delta^1_{1j})\frac{\partial}{\partial y_1} + (-\varepsilon_j y_1 + \bar\delta^j_{1j})\frac{\partial}{\partial y_j} + \sum_{i=2\, i\neq j}^{m} \bar\delta^i_{1j}\frac{\partial}{\partial y_i} - \varepsilon_j(w_j+\eta_1)\frac{\partial}{\partial w_j}, \\
&(\varepsilon_1 y_l + \bar\delta^1_{1l})\frac{\partial}{\partial y_1} + (-\varepsilon_l y_1 + \bar\delta^l_{1l})\frac{\partial}{\partial y_l} + \sum_{k=2\, k\neq l}^{m} \bar\delta^k_{1l}\frac{\partial}{\partial y_k} - \varepsilon_l(w_l+\eta_1)\frac{\partial}{\partial w_l} \Bigg] \\
= &\left(\varepsilon_l(w_1+\eta_1)\frac{\partial \bar\delta^1_{1j}}{\partial w_l} - \varepsilon_j(w_1+\eta_1)\frac{\partial \bar\delta^1_{1l}}{\partial w_j} + \varepsilon_1\bar\delta^l_{1j} - \varepsilon_1\bar\delta^j_{1l} \right)\frac{\partial}{\partial y_1} \\
+ &\left(\varepsilon_l(w_1+\eta_1)\frac{\partial \bar\delta^j_{1j}}{\partial w_l} - \varepsilon_j(w_1+\eta_1)\frac{\partial \bar\delta^j_{1l}}{\partial w_j} + \varepsilon_j(\varepsilon_1 y_l + \bar\delta^1_{1l}) \right)\frac{\partial}{\partial y_j} \\
+ &\left(\varepsilon_l(w_1+\eta_1)\frac{\partial \bar\delta^l_{1j}}{\partial w_l} - \varepsilon_j(w_1+\eta_1)\frac{\partial \bar\delta^l_{1l}}{\partial w_j} - \varepsilon_l(\varepsilon_1 y_j + \bar\delta^1_{1j}) \right)\frac{\partial}{\partial y_l} \\
+ &\sum_{i=2,\, i\neq j,\, i\neq l}^{m} \left(\varepsilon_l(w_1+\eta_1)\frac{\partial \bar\delta^i_{1j}}{\partial w_l} - \varepsilon_j(w_1+\eta_1)\frac{\partial \bar\delta^i_{1l}}{\partial w_j} \right)\frac{\partial}{\partial y_i}
\end{aligned}$$

Thus we have $[\Delta_{1j}, \Delta_{1l}] \notin \Delta$, since there are no components along $\frac{\partial}{\partial w}$, a contradiction.

We have therefore showed that relation (6.9) holds in any case ($m = 1$, $m = 2$, and $m \geq 3$) and the proof is complete. ■

The proof is interesting as it is constructive: we have not only transformed Σ into $\Sigma^{0,\eta}_{p,q}$ but we have also transformed the elements of the basis of $\mathfrak{L}$ into the corresponding elements of the basis of $\mathfrak{L}^0_{p,q}$ given by (6.2).

Remark. In the case $m = 1$, notice that the proof is shorten as $\mathfrak{so}(1) = \{0\}$ thus the proof stops at equation (6.7') (in the case $f(0) \in \mathcal{D}^0(0)$). Indeed this equation is sufficient to conclude that $f^0(w) = \lambda w^2$ because it is the only scalar function which is homogeneous of degree 2. ◆

3 Conclusion and Perspectives

In this chapter we studied the characterisation of null-forms of paraboloid systems $\Sigma^0_{p,q}$ via their Lie algebra of infinitesimal symmetries. We showed that this class of control-affine systems is completely determined by its Lie algebra of symmetries (under some regularity assumption that are necessary as $\Sigma^0_{p,q}$ satisfies them). We believe that there are very few results of that kind for control-affine systems existing in the literature, with a notable exception [DZ14], and more for the control-linear case, see [AK11; DK14; Kru12]. In view of all those results, it would be very interesting to study the following generalisation.

Problem. Let $\mathfrak{L}$ be a finite-dimensional Lie algebra acting transitively on a manifold $\mathcal{M}$. Does this Lie algebra uniquely determine a class of control-affine systems (around a generic point) by being isomorphic to its algebra of infinitesimal symmetries.

We believe that considering control-affine systems instead of control-linear system in that problem matters. A reason is that the existence of the drift f gives more rigidity on the symmetries and therefore would constraint the structure of a system possessing those symmetries. To start investigating this problem, we could begin with the study of all 2- and 3-dimensional Lie algebras (which are well known) and to characterise the control-affine systems that admit those Lie algebras as symmetries. A starting point is Theorem 4.1 in our paper [SR21], where we assert that 3 model of 3-dimensional Lie algebras completely characterise different families of control-affine systems. It would also be interesting to allow changes in the time parametrisation of control systems (i.e. add the equation $\dot{t} = 1$) and thus to find analogous results as those presented in [ANN15].

Another research axis would be the study of the possible correspondence between the symmetries of second prolongations of submanifolds (as used in this chapter) and the classical symmetries of underdetermined implicit differential equations, see e.g. [Olv86; Olv95; Ibr93] for an introduction on this topic. Preliminary results show that a connection should be possible but is complicate to be establish as our parametrisations are implicit in w. A starting point would be the study for the class of differential equations of the form $\{\dot{z} - s(x, \dot{y}) = 0\}$.

6.A Lie algebra of symmetries of $\Sigma^0_{p,q}$

In this appendix, we show how to compute $\mathfrak{L}^0_{p,q}$, the Lie algebra of infinitesimal symmetries of $\Sigma^0_{p,q}$. Consider a null-form (p,q)-paraboloid system $\Sigma^0_{p,q} = (f,g)$ given by the vector fields $f = \sum_{i=1}^m \varepsilon_i (w_i)^2 \frac{\partial}{\partial z} + \sum_{i=1}^m w_i \frac{\partial}{\partial y_i}$ and $g_i = \frac{\partial}{\partial w_i}$. Let $v = v^0(z,y,w)\frac{\partial}{\partial z} + \sum_{i=1}^m v_1^i(z,y,w)\frac{\partial}{\partial y_i} + v_2^i(z,y,w)\frac{\partial}{\partial w_i}$ be an infinitesimal symmetry of $\Sigma^0_{p,}$. Then, using Proposition 1.4 of Chapter 1 we deduce that

$$[g_i, v] \in \mathcal{D}^0 \quad \Rightarrow \quad v^0 = v^0(z,y) \quad \text{and} \quad v_1^i = v_1^i(z,y), \quad \text{for} \quad 1 \le i \le m.$$

Moreover, using $[f,v] \in \mathcal{D}^0$ we obtain that

$$\left[\sum_{i=1}^m \varepsilon_i (w_i)^2 \frac{\partial}{\partial z} + w_i \frac{\partial}{\partial y_i}, v^0 \frac{\partial}{\partial z} + \sum_{j=1}^m \left(v_1^j \frac{\partial}{\partial y_j} + v_2^j \frac{\partial}{\partial w_j}\right)\right] = 0 \quad \mathrm{mod}\ \mathcal{D}^0$$
$$\sum_{i=1}^m \left(\varepsilon_i (w_i)^2 \frac{\partial v^0}{\partial z} + w_i \frac{\partial v^0}{\partial y_i} - 2\varepsilon_i w_i v_2^i \right) \frac{\partial}{\partial z}$$
$$+ \sum_{j=1}^m \left[\sum_{i=1}^m \left(\varepsilon_i (w_i)^2 \frac{\partial v_1^j}{\partial z} + w_i \frac{\partial v_1^j}{\partial y_i}\right) - v_2^j \right] \frac{\partial}{\partial y_j} = 0,$$

which yields

$$\text{(6.12)} \qquad \begin{cases} \sum_{i=1}^m \varepsilon_i (w_i)^2 \frac{\partial v^0}{\partial z} + w_i \frac{\partial v^0}{\partial y_i} & = 2\sum_{i=1}^m \varepsilon_i w_i v_2^i, \\ \sum_{i=1}^m \left(\varepsilon_i (w_i)^2 \frac{\partial v_1^j}{\partial z} + w_i \frac{\partial v_1^j}{\partial y_i}\right) & = v_2^j, \quad 1 \le j \le m. \end{cases}$$

By inserting the second equation of (6.12) in the first we obtain

$$\sum_{i=1}^m \varepsilon_i (w_i)^2 \frac{\partial v^0}{\partial z} + w_i \frac{\partial v^0}{\partial y_i} = 2 \sum_{i=1}^m \varepsilon_i w_i \sum_{k=1}^m \left(\varepsilon_k (w_k)^2 \frac{\partial v_1^i}{\partial z} + w_k \frac{\partial v_1^i}{\partial y_k}\right)$$

and equating the terms of the same degree in w_i we deduce the conditions

$$\text{(6.13)} \qquad \frac{\partial v_1^i}{\partial z} = 0, \quad \frac{\partial v^0}{\partial y_i} = 0, \quad \varepsilon_i \frac{\partial v^0}{\partial z} = 2\varepsilon_i \frac{\partial v_1^i}{\partial y_i}, \quad 2\varepsilon_i \frac{\partial v_1^i}{\partial y_k} + 2\varepsilon_k \frac{\partial v_1^k}{\partial y_i} = 0,\ i \neq k.$$

Solving the first three conditions implies that $v^0 = v^0(z) = 2az + b_0$ with $a, b_0 \in \mathbb{R}$, and $v_1^i = ay_i + \hat{v}_1^i(y_1, \ldots, \cancel{y_i}, \ldots, y_m)$. Then, using the last condition of (6.13) we get

$$\text{for all } i \neq k, \quad \varepsilon_i \frac{\partial \hat{v}_1^i}{\partial y_k} + \varepsilon_k \frac{\partial \hat{v}_1^k}{\partial y_i} = 0 \Rightarrow \varepsilon_i \frac{\partial^2 \hat{v}_1^i}{\partial y_k^2} = 0 \Rightarrow \hat{v}_1^i = \sum_{j \neq i} b_j^i y_j + c^i,$$

where $b_j^i, c^i \in \mathbb{R}$ satisfy $\varepsilon_i b_k^i + \varepsilon_k b_i^k = 0$. And finally, using the second equation of (6.12), we have

$$v_2^i = \sum_{j=1}^m w_j \frac{\partial v_1^i}{\partial y_j} = aw_i + \sum_{j \neq i} b_j^i w_j.$$

So if $v \in \mathfrak{L}^0_{p,q}$, then it can be written as

$$(6.14)\quad v = (2a+b_0)\frac{\partial}{\partial z} + \sum_{i=1}^{m}\left(ay_i + \sum_{j\neq i} b^i_j y_j + c^i\right)\frac{\partial}{\partial y_i} + \sum_{i=1}^{m}\left(aw_i + \sum_{j\neq i} b^i_j w_j\right)\frac{\partial}{\partial w_i},$$

where a, b_0, c^i, b^i_j are real constants satisfying $\varepsilon_i b^i_j + \varepsilon_j b^j_i = 0$ for all $i \neq j$. Observe that we have a sort of skew symmetry of the matrix $B := (b^i_j)$ twisted by the $\varepsilon_i = \pm 1$, so knowing half of the matrix $1 \leq i < j \leq m$ is sufficient to reconstruct B completely. Therefore, choosing special values of the constant, we obtain the following basis vectors of $\mathfrak{L}^0_{p,q}$:

$$\begin{aligned}
&b_0 = 1,\ a = c^i = b^i_j = 0, \quad \frac{\partial}{\partial z} \in \mathfrak{L}^0_{p,q},\\
&c^i = 1,\ a = b_0 = b^i_j = 0, \quad \frac{\partial}{\partial y_i} \in \mathfrak{L}^0_{p,q}, \quad \text{for} \quad 1 \leq i \leq m,\\
&a = 1,\ b_0 = c^i = b^i_j = 0, \quad 2z\frac{\partial}{\partial z} + \sum_{i=1}^{m} y_i\frac{\partial}{\partial y_i} + w_i\frac{\partial}{\partial w_i} \in \mathfrak{L}^0_{p,q},\\
&a = b_0 = c^i = 0,\ b^i_j = 0 \quad \text{except for} \quad b^k_l = \varepsilon_k \quad \text{implying that} \quad b^l_k = -\varepsilon_l,\\
&\qquad \text{so} \qquad \varepsilon_k y_l\frac{\partial}{\partial y_k} - \varepsilon_l y_k\frac{\partial}{\partial y_l} + \varepsilon_k w_l\frac{\partial}{\partial w_k} - \varepsilon_l w_k\frac{\partial}{\partial w_l} \in \mathfrak{L}^0_{p,q}, \quad \text{for} \quad 1 \leq k < l \leq m,
\end{aligned}$$

and there are no difficulties to show that any v as in (6.14) can be written as a linear combination of the above vector fields. Therefore,

$$\begin{aligned}
\mathfrak{L}^0_{p,q} = \operatorname{vect}_{\mathbb{R}}\left\{\frac{\partial}{\partial z}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_m}\right\} \oplus \operatorname{vect}_{\mathbb{R}}\left\{2z\frac{\partial}{\partial z} + \sum_{i=1}^{m} y_i\frac{\partial}{\partial y_i} + w_i\frac{\partial}{\partial w_i}\right\}\\
\oplus \operatorname{vect}_{\mathbb{R}}\left\{\varepsilon_k y_l\frac{\partial}{\partial y_k} - \varepsilon_l y_k\frac{\partial}{\partial y_l} + \varepsilon_k w_l\frac{\partial}{\partial w_k} - \varepsilon_l w_k\frac{\partial}{\partial w_l},\ 1 \leq k < l \leq m\right\}.
\end{aligned}$$

6.B Commutativity relations of $\mathfrak{L}^0_{pq}$

In this appendix, we detail the computation of the commutativity relations of $\mathfrak{L}^0_{p,q}$. Recall the following notations:

$$\begin{aligned}
v_0 = \frac{\partial}{\partial z}, \quad v_i = \frac{\partial}{\partial y_i} \quad \text{for} \quad 1 \leq i_= \leq m, \quad E = 2z\frac{\partial}{\partial z} + \sum_{i=1}^{m} y_i\frac{\partial}{\partial y_i} + w_i\frac{\partial}{\partial w_i},\\
\Delta_{kl} = \varepsilon_k y_l\frac{\partial}{\partial y_k} - \varepsilon_l y_k\frac{\partial}{\partial y_l} + \varepsilon_k w_l\frac{\partial}{\partial w_k} - \varepsilon_l w_k\frac{\partial}{\partial w_l}, \quad \text{for} \quad 1 \leq k < l \leq m.
\end{aligned}$$

We obviously have $[v_0, v_i] = 0$, $[v_i, v_j] = 0$, and $[v_0, \Delta_{kl}] = 0$. By a straightforward computation, we deduce that $[v_0, E] = 2v_0$ and that for all $1 \leq i \leq m$ we have

$[v_i, E] = v_i$. Next, we obtain

$$[v_i, \Delta_{kl}] = \left[\frac{\partial}{\partial y_i}, \varepsilon_k y_l \frac{\partial}{\partial y_k} - \varepsilon_l y_k \frac{\partial}{\partial y_l}\right] = \begin{cases} \varepsilon_k \frac{\partial}{\partial y_k} = \varepsilon_k v_k & \text{if} \quad i = l, \\ -\varepsilon_l \frac{\partial}{\partial y_l} = -\varepsilon_l v_l & \text{if} \quad i = l, \\ 0 & \text{otherwise,} \end{cases}$$
$$= \varepsilon_k \delta_i^l v_k - \varepsilon_l \delta_i^k v_l,$$

where δ_i^l denotes the Kronecker delta operator.

$$\begin{aligned} [E, \Delta_{kl}] &= \left[2z\frac{\partial}{\partial z} + \sum_{i=1}^{m} y_i \frac{\partial}{\partial y_i} + w_i \frac{\partial}{\partial w_i}, \varepsilon_k y_l \frac{\partial}{\partial y_k} - \varepsilon_l y_k \frac{\partial}{\partial y_l} + \varepsilon_k w_l \frac{\partial}{\partial w_k} - \varepsilon_l w_k \frac{\partial}{\partial w_l}\right] \\ &= \sum_{i=1}^{m} \left[y_i \frac{\partial}{\partial y_i}, \varepsilon_k y_l \frac{\partial}{\partial y_k} - \varepsilon_l y_k \frac{\partial}{\partial y_l}\right] + \left[w_i \frac{\partial}{\partial w_i}, \varepsilon_k w_l \frac{\partial}{\partial w_k} - \varepsilon_l w_k \frac{\partial}{\partial w_l}\right] \\ &= \underbrace{\varepsilon_k y_l \frac{\partial}{\partial y_k} + \varepsilon_l y_k \frac{\partial}{\partial y_l}}_{i=l} - \underbrace{\varepsilon_k y_l \frac{\partial}{\partial y_k} - \varepsilon_l y_k \frac{\partial}{\partial y_l}}_{i=k} \\ &\qquad + \underbrace{\varepsilon_k w_l \frac{\partial}{\partial w_k} + \varepsilon_l w_k \frac{\partial}{\partial w_l}}_{i=l} - \underbrace{\varepsilon_k w_l \frac{\partial}{\partial w_k} - \varepsilon_l w_k \frac{\partial}{\partial w_l}}_{i=k} \\ &= 0. \end{aligned}$$

And finally, for the brackets between the vector fields Δ_{kl}, we develop the computations along the $\frac{\partial}{\partial y_i}$ components only because it is the same along the $\frac{\partial}{\partial w_i}$ components. Hence, we have

$$\begin{aligned} [\Delta_{ij}, \Delta_{kl}] &= \left[\varepsilon_i y_j \frac{\partial}{\partial y_i} - \varepsilon_j y_i \frac{\partial}{\partial y_j}, \varepsilon_k y_l \frac{\partial}{\partial y_k} - \varepsilon_l y_k \frac{\partial}{\partial y_l}\right] \\ &= \delta_i^k \left(-\varepsilon_i \varepsilon_l y_j \frac{\partial}{\partial y_l} + \varepsilon_i \varepsilon_j y_l \frac{\partial}{\partial y_j}\right) + \delta_i^l \left(-\varepsilon_i \varepsilon_l y_k \frac{\partial}{\partial y_j} + \varepsilon_i \varepsilon_k y_j \frac{\partial}{\partial y_k}\right) \\ &\quad + \delta_j^k \left(-\varepsilon_i \varepsilon_j y_l \frac{\partial}{\partial y_i} + \varepsilon_j \varepsilon_l y_i \frac{\partial}{\partial y_j}\right) + \delta_j^l \left(-\varepsilon_j \varepsilon_k y_i \frac{\partial}{\partial y_k} + \varepsilon_i \varepsilon_j y_k \frac{\partial}{\partial y_i}\right) \\ &= -\varepsilon_i \delta_i^k \Delta_{jl} + \varepsilon_i \delta_i^l \Delta_{jk} + \varepsilon_j \delta_j^k \Delta_{il} - \varepsilon_j \delta_j^l \Delta_{ik}. \end{aligned}$$

Therefore $[\Delta_{ij}, \Delta_{kl}] \neq 0$ if and only if either $i = k$, or $i = l$, or $j = k$, or $j = l$. In the first case, $[\Delta_{ij}, \Delta_{il}] = -\varepsilon_i \Delta_{jl}$ for $1 \leq i < j < l \leq m$ and all other cases can be deduced by permutations of the indices in that formula, recall that we define $\Delta_{lk} = -\Delta_{kl}$.

6.C Description of $\mathfrak{so}(p, q)$

In this appendix, we recall some general facts about the Lie algebra $\mathfrak{so}(p, q)$. We first recall the definition of the indefinite orthogonal group $O(p, q)$. Let p and q be positive integers, with $p \geq q$, and consider $\mathbb{R}^m = \mathbb{R}^{p+q}$. Define a symmetric bilinear form $(\cdot, \cdot)_{p,q}$ on $\mathbb{R}^m$ by the formula

$$(x, y)_{p,q} = x_1 y_1 + \ldots + x_p y_p - x_{p+1} y_{p+1} - \ldots - x_m y_m.$$

The set of real matrices A of dimension $m \times m$ which preserve this form, i.e. $(Ax, Ay)_{p,q} = (x, y)_{p,q}$ for all $x, y \in \mathbb{R}^m$, is called the *generalised orthogonal group* $O(p, q)$. Let $\mathtt{I}_{p,q}$ denote the $m \times m$ diagonal matrix with ones in the first p diagonal entries and minus ones in the last q diagonal entries. The,n A is in $O(p, q)$ if and only if $A^t \mathtt{I}_{p,q} A = \mathtt{I}_{p,q}$. Taking the determinant of the last relation gives $(\det A)^2 \det \mathtt{I}_{p,q} = \det \mathtt{I}_{p,q}$. Thus for any $A \in O(p, q)$, we have $\det A = \pm 1$.

Recall that the elements X of the Lie algebra $\mathfrak{g}$ (of a Lie group G) are characterised by the property that $e^{tX} \in G$ for all $t \geq 0$. Therefore, if X is a $m \times m$ real matrix, then $e^{tX} \in O(p, q)$ if and only if

$$e^{tX^t} \mathtt{I}_{p,q} e^{tX} = \mathtt{I}_{p,q} \Rightarrow \mathtt{I}_{p,q} e^{tX^t} \mathtt{I}_{p,q} = e^{-tX} \Rightarrow e^{t \mathtt{I}_{p,q} X^t \mathtt{I}_{p,q}} = e^{-tX}.$$

This condition holds for all real $t \geq 0$ if and only if $\mathtt{I}_{p,q} X^t \mathtt{I}_{p,q} = -X$ or, equivalently, $X^t \mathtt{I}_{p,q} + X \mathtt{I}_{p,q} = 0$. Thus the Lie algebra of $O(p, q)$ consists of all $m \times m$ matrices X satisfying $X^t \mathtt{I}_{p,q} + X \mathtt{I}_{p,q} = 0$. This Lie algebra is denoted $\mathfrak{so}(p, q)$.

The Lie algebra $\mathfrak{so}(p, q)$, see [Ada12] for further details and examples, can be embedded in $\mathfrak{gl}_m = \mathrm{Mat}(m)$ as follows. We use the block decomposition $X = \left(\begin{smallmatrix} X_1 & X_2 \\ X_3 & X_4 \end{smallmatrix}\right)$, thus applying $X^t \mathtt{I}_{p,q} + X \mathtt{I}_{p,q} = 0$ we obtain

$$X_1^t + X_1 = 0, \quad X_2 - X_3^t = 0, \quad \text{and} \quad X_4^t + X_4 = 0.$$

Hence, we get $X = \begin{pmatrix} X_1 & X_2 \\ X_2^t & X_4 \end{pmatrix}$ with X_1 and X_4 being skew symmetric matrices of dimension $p \times p$ and $q \times q$ respectively, and X_2 is an arbitrary $p \times q$ matrix. So a basis of $\mathfrak{so}(p, q)$ is given by

$$\begin{aligned} S_{kl} &= \begin{cases} E_{kl} - E_{lk} & \text{if} \quad 1 \leq k < l \leq p, \\ E_{kl} + E_{lk} & \text{if} \quad 1 \leq k \leq p \text{ and } p+1 \leq l \leq m, \\ E_{lk} - E_{kl} & \text{if} \quad p+1 \leq k < l \leq m \end{cases}, \\ &= \varepsilon_k E_{kl} - \varepsilon_l E_{lk}, \quad \forall\, 1 \leq k < l \leq m, \end{aligned}$$

where E_{kl} is the elementary matrix with all entries being zeroes except for a 1 in the k-th line and l-th column; and recall that $\varepsilon_i = \begin{cases} +1 & 1 \leq i \leq p \\ -1 & p+1 \leq i \leq m \end{cases}$. This basis of $\mathfrak{so}(p, q)$ shows that $\dim \mathfrak{so}(p, q) = \frac{m(m-1)}{2}$. Finally, we show that the Lie algebra of vector fields

$$\Delta = \mathrm{vect}_{\mathbb{R}} \left\{ \Delta_{kl} = \varepsilon_k y_l \frac{\partial}{\partial y_k} - \varepsilon_l y_k \frac{\partial}{\partial y_l} + \varepsilon_k w_l \frac{\partial}{\partial w_k} - \varepsilon_l w_k \frac{\partial}{\partial w_l},\ 1 \leq k < l \leq m \right\}$$

is isomorphic to $\mathfrak{so}(p, q)$. First, observe that the two Lie algebras are of the same dimension. Second, it is clear that the The following map

$$\rho \,:\, y_l \frac{\partial}{\partial y_k} + w_l \frac{\partial}{\partial w_k} \mapsto E_{kl},$$

is an isomorphism between the basis elements Δ_{kl} of Δ and the basis elements S_{kl} of $\mathfrak{so}(p, q)$.

Appendix A

An extension of Frobenius theorem

In this appendix, we present a generalisation of the Frobenius theorem on the rectification of constant rank involutive distributions whose usual version can be stated as

Theorem A.1 (Frobenius theorem)**.** *On a smooth n-dimensional manifold $\mathcal{X}$, we consider a distribution $\mathcal{D}$. There exists local coordinates $(x^1, \dots, x^d, y^1, \dots, y^{n-d})$ such that*

$$\mathcal{D} = \text{span}\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^d}\right\},$$

if and only if $\mathcal{D}$ is involutive and of constant rank d.

For a proof see [Lun92]. Our generalisation gives necessary and sufficient conditions for the simultaneous rectification of m involutive distributions of constant rank. We consider $\mathcal{X}$ a smooth n-dimensional manifold equipped with local coordinates ξ; and m involutive distributions $\mathcal{D}_1, \dots, \mathcal{D}_m$ of constant rank $\text{rk}(\mathcal{D}_i) = d_i$ satisfying

$$\dim\left(\mathcal{D}_1(x_0) \oplus \cdots \oplus \mathcal{D}_m(x_0)\right) = d, \quad \text{where} \quad d := \sum_{i=1}^{n} d_i \leq N.$$

The above condition means that the vector spaces $\mathcal{D}_i(x_0)$ are complementary, i.e. $\mathcal{D}_i(x_0) \cap \mathcal{D}_j(x_0) = 0$ for all $i \neq j$. We denote by $\mathcal{D}_{i_1,\dots,i_k}$, for $k \leq m$, the distribution $\mathcal{D}_{i_1} \oplus \cdots \oplus \mathcal{D}_{i_k}$, with the convention that $\mathcal{D}_{i,i} = \mathcal{D}_i$.

Theorem A.2 (Generalisation of Frobenius Theorem)**.** *Let $\mathcal{D}_1, \dots, \mathcal{D}_m$ be m distributions of constant rank and satisfying $\mathcal{D}_i(x_0) \cap \mathcal{D}_j(x_0) = 0$ for all $i \neq j$. Then, locally around x_0, there exists a coordinate system $\left(x_1^1, \dots, x_1^{d_1}, \dots, x_n^1, \dots, x_n^{d_n}, y^1, \dots, y^{N-d}\right)$ such that*

$$\mathcal{D}_i = span\left\{\frac{\partial}{\partial x_i^1}, \dots, \frac{\partial}{\partial x_i^{d_i}}\right\}, \quad \textit{for all} \quad 1 \leq i \leq m,$$

if and only if for all indices $i_1, i_2 \in \{1, \dots, m\}$ the distribution $\mathcal{D}_{i_1,i_2}$ is involutive.

Clearly our result generalises Frobenius theorem since for $m = 1$ our necessary and sufficient condition reads $\mathcal{D}_1$ is involutive and of constant rank. We mention that different (although equivalent) conditions for that result have been obtained in [Res82, Lemma 3.1]. Our proof will use the following lemma which shows that our assumptions imply that the sum of any number of distributions $\mathcal{D}_i$ is involutive.

Lemma 1.1. *Under the above assumptions, if for all indices $i_1, i_2 \in \{1, \ldots, n\}$ the distributions $\mathcal{D}_{i_1,i_2}$ are involutive, then for any multi-index $i_1, \ldots, i_k$, with $k \leq n$, the distribution $\mathcal{D}_{i_1,\ldots,i_k}$ is involutive.*

Proof. By a direct computation. For any $v, w \in \mathcal{D}_{i_1,\ldots,i_k}$ we have $v = v_{i_1} + \ldots + v_{i_k}$ with $v_{i_j} \in \mathcal{D}_{i_j}$ (similarly for w) and thus

$$[v, w] = \left[\sum_{j=1}^{k} v_{i_j}, \sum_{l=1}^{k} w_{i_l}\right] = \sum_{j=1}^{k}\sum_{l=1}^{k}\left[v_{i_j}, w_{i_l}\right].$$

Since $\mathcal{D}_{i_j,i_l}$ is involutive for every i_j and i_l, the brackets of the right hand side are in $\mathcal{D}_{i_j,i_l} \subset \mathcal{D}_{i_1,\ldots,i_k}$ and the conclusion follows. ∎

We will also need the following lemma that is a special case of our general result stated for $m = 2$ distributions.

Lemma 1.2. *Consider $\mathcal{A}$ and $\mathcal{B}$ two involutive distributions of constant rank a and b, respectively. Suppose that $\mathcal{A} \cap \mathcal{B} = \{0\}$ and that $\mathcal{A} \oplus \mathcal{B}$ is involutive. Let $y^0 = (y^0_1, \ldots, y^0_c)$, where $c = n-(a+b)$, be local functions such that $\mathcal{A} \oplus \mathcal{B} = \ker \mathrm{d}y^0$. Then, we can complete them to local coordinates (y^0, y^1, y^2), where $y^1 = (y^1_1, \ldots, y^1_a)$ and $y^2 = (y^2_1, \ldots, y^2_b)$, such that*

$$\mathcal{A} = \operatorname{span}\left\{\frac{\partial}{\partial y^1_1}, \ldots, \frac{\partial}{\partial y^1_a}\right\} = \ker \mathrm{d}y^0 \bigcap \ker \mathrm{d}y^2, \quad \textit{and}$$

$$\mathcal{B} = \operatorname{span}\left\{\frac{\partial}{\partial y^2_1}, \ldots, \frac{\partial}{\partial y^2_b}\right\} = \ker \mathrm{d}y^0 \bigcap \ker \mathrm{d}y^1.$$

Proof. Choose (locally) vector fields A_i, for $1 \leq i \leq a$, and B_j, for $1 \leq j \leq b$, such that $\mathcal{A} = \operatorname{span}\{A_1, \ldots, A_a\}$ and $\mathcal{B} = \operatorname{span}\{B_1, \ldots, B_b\}$, and complete them by C_k, for $1 \leq k \leq c$, such that the n vector fields A_i, B_j, and C_k are independent. Given a coordinate system, form the matrices $A = (A_1, \ldots, A_a)$, $B = (B_1, \ldots, B_b)$, $C = (C_1, \ldots, C_c)$, and for a local $\mathbb{R}^k$-valued function ϕ form the matrix $\mathrm{d}\phi \cdot A = \langle \mathrm{d}\phi, A\rangle = (\mathrm{d}\phi^i \cdot A_j)$ and similarly for B and C.

The distribution $\mathcal{A} \oplus \mathcal{B}$ is involutive and of constant rank, so we can find local coordinates $y^0 = (y^0_1, \ldots, y^0_c)$ such that $\mathrm{d}y^0 \cdot A = 0$ and $\mathrm{d}y^0 \cdot B = 0$. We set $D^0 = \mathrm{d}y^0 C$ and we have $\operatorname{rk} D^0 = c$. The distribution $\mathcal{A}$ is involutive, so we can complete y^0 by $\hat{y}^1 = (\hat{y}^1_1, \ldots, \hat{y}^1_a)$ and $y^2 = (y^2_1, \ldots, y^2_b)$ such that $\operatorname{ann}(\mathcal{A}) = \operatorname{span}\{\mathrm{d}y^2\}$, i.e. $\mathrm{d}y^2 \cdot A = 0$. Moreover, by $\mathrm{d}y^0 \wedge \mathrm{d}\hat{y}^1 \wedge \mathrm{d}y^2 \neq 0$, it follows that $\operatorname{rk} \hat{D}^1 = a$ and $\operatorname{rk} D^2 = b$, where $\hat{D}^1 = \mathrm{d}\hat{y}^1 \cdot A$ and $D^2 = \mathrm{d}y^2 \cdot B$. Similarly, the involutivity of $\mathcal{B}$ yields the existence of $y^1 = (y^1_1, \ldots, y^1_a)$ and $\hat{y}^2 = (\hat{y}^2_1, \ldots, \hat{y}^2_b)$ such that $\operatorname{ann}(\mathcal{B}) = \operatorname{span}\{\mathrm{d}y^1\}$. If follows that $\operatorname{rk} D^1 = a$ and $\operatorname{rk} \hat{D}^2 = b$, where $D^1 = \mathrm{d}y^1 \cdot A$ and $\hat{D}^2 = \mathrm{d}\hat{y}^2 \cdot B$. To summarise, we have

$$\begin{array}{cccc} & C & A & B \\ \mathrm{d}y^0 & D^0 & 0 & 0 \\ \mathrm{d}\hat{y}^1 & * & \hat{D}^1 & * \\ \mathrm{d}y^2 & * & 0 & D^2 \end{array}, \quad \text{and} \quad \begin{array}{cccc} & C & A & B \\ \mathrm{d}y^0 & D^0 & 0 & 0 \\ \mathrm{d}y^1 & * & D^1 & 0 \\ \mathrm{d}\hat{y}^2 & * & * & \hat{D}^2 \end{array}.$$

Now we claim that the functions y^0, y^1, y^2 form a local coordinate system such that $\mathcal{A} = \ker \mathrm{d}y^0 \cap \ker \mathrm{d}y^2$ and $\mathcal{B} = \ker \mathrm{d}y^0 \cap \ker \mathrm{d}y^1$. The former and the latter statement follow immediately from

$$\begin{array}{cccc} & C & A & B \\ \mathrm{d}y^0 & D^0 & 0 & 0 \\ \mathrm{d}y^1 & * & D^1 & 0 \\ \mathrm{d}y^2 & * & 0 & D^2 \end{array}$$

and from $\operatorname{rk} D^1 = a$ and $\operatorname{rk} D^2 = b$. ■

We can now prove our generalisation of Frobenius theorem.

Proof. Assume that $n \geq 2$ (otherwise it is just the classical Frobenius theorem), we proceed by induction. Apply Lemma 1.2 to $\mathcal{A} = \mathcal{D}_1$ and $\mathcal{B} = \mathcal{D}_{2,\ldots,m}$ with y^0 such that $\mathrm{d}y^0 \in \operatorname{ann}(\mathcal{D}_{1,\ldots,m})$; notice that $\mathcal{B}$ is involutive by Lemma 1.1. There thus exists local coordinates (y^0, y^1, y^2), that we rename $x^0 = y^0$ and $x^1 = y^1$, such that $\mathcal{D}_{1,\ldots,m} = \ker \mathrm{d}x^0$, $\mathcal{D}_1 = \operatorname{span}\left\{\frac{\partial}{\partial x^1}\right\}$, and $\mathcal{D}_{2,\ldots,n} = \ker \mathrm{d}x^0 \cap \ker \mathrm{d}x^1$. Now, let $k \geq 2$ and assume that we locally have produced functions $(x^0, \ldots, x^{k-1})$ such that $\mathcal{D}_i = \operatorname{span}\left\{\frac{\partial}{\partial x^i}\right\}$, for $1 \leq i \leq k-1$, and $\mathrm{d}x^j \in \operatorname{ann}(\mathcal{D}_{k,\ldots,m})$ for $0 \leq j \leq k-1$. Apply Lemma 1.2 to $\mathcal{A} = \mathcal{D}_k$ and $\mathcal{B} = \mathcal{D}_{k+1,\ldots,m}$, where $y^0 = (x^0, \ldots, x^{k-1})$ to get (y^0, y^1, y^2), where we set $x^0 = y^0$ and $x^k = y^1$, such that $\mathcal{D}_k = \operatorname{span}\left\{\frac{\partial}{\partial x^k}\right\}$ and $\mathcal{D}_{k+1,\ldots,m} = \ker \mathrm{d}x^0 \cap \ker \mathrm{d}x^k$; so for all $1 \leq i \leq k$ we get $\mathcal{D}_i = \operatorname{span}\left\{\frac{\partial}{\partial x^i}\right\}$. After m applications of Lemma 1.2 we get the desired form of $\mathcal{D}_1, \ldots, \mathcal{D}_m$. ■

We terminate this appendix with the following corollary that we use several times in the thesis.

Corollary A.1. *Assume that $\mathcal{X}$ is 2-dimensional and let A and B be two smooth vector fields on $\mathcal{X}$ satisfying $A \wedge B \neq 0$. Then, there exists a local coordinate system $x = (x^1, x^2)$ such that $A = a(x)\frac{\partial}{\partial x^1}$ and $B = b(x)\frac{\partial}{\partial x^2}$, where a and b are smooth functions satisfying $ab \neq 0$.*

Part II

Motion planning for control systems

Chapter 7

Controllability and motion planning review

The second part of this manuscript is devoted to the problem of motion planning of control systems. This is the problem of finding algorithms that compute controls such that the system is brought into a prescribed target configuration. Motion planning has obvious applications to robotics, autonomous vehicles, aerospace and, in general, to any domain where dynamical systems with controls are involved. Throughout the last thirty years several methods have been proposed, and in this thesis we are particularly interested in the *continuation method* that was introduced by Chitour and Sussmann. That method relies on solving a highly nonlinear differential equation in an infinite-dimensional space, which might be ill-posed due to singularities. We propose a regularisation of this method that overcomes that difficulty. In the next chapter, we will show that when the regularising parameter goes to zero, we get a solution of the original problem and we will illustrate our method through several numerical examples.

This chapter is organised as follows. In the next section, we give some definitions and notations, and we introduce the problem of motion planning. We also review some properties about the controllability of control systems and about the endpoint mapping. In section 2, we present several existing algorithms and methodology for solving the motion planning problem.

1 Preliminaries

Throughout this part of the manuscript, the word *smooth* will always mean C^∞-smooth, manifolds are «smooth, finite-dimensional, Hausdorff, second countable, and paracompact», and all objects (vector fields, tensor fields, functions) are considered smooth. For a manifold $\mathcal{X}$ we will denote by $T\mathcal{X}$ and $T^*\mathcal{X}$ the tangent and cotangent bundle, respectively. The space of all smooth vector fields (smooth sections of $T\mathcal{X}$) will be denoted $V^\infty(\mathcal{X})$ and the space of all smooth differential p-forms by $\Lambda^p(\mathcal{X})$, except for smooth functions (0-forms) whose space is denoted $C^\infty(\mathcal{X})$. On a smooth n-dimensional manifold $\mathcal{X}$ (or an open subset of $\mathbb{R}^n$), we consider a control-affine system, with state $x \in \mathcal{X}$ and $m \geq 1$ controls, given in local coordinates by

$$\Sigma \,:\, \dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x), \quad \text{and} \quad x(0) = x_0.$$

The control $u = (u_1, \ldots, u_m)$ takes values in U, an open connected subset of $\mathbb{R}^m$ containing 0, and $f, g_1, \ldots, g_m$ are smooth vector fields on $\mathcal{X}$. Throughout this part of the manuscript, we assume that the initial state $x_0 \in \mathcal{X}$ and the final time $T > 0$ are fixed. A trajectory of Σ is a smooth curve $\gamma \,:\, [0,T] \to \mathcal{X}$ such that $\gamma(0) = x_0$ and that there exists a control $u(\cdot) \in U$ for which

$$\dot{\gamma}(t) = f(\gamma(t)) + \sum_{i=1}^{m} u_i(t) g_i(\gamma(t))$$

holds almost everywhere on $[0,T]$. Since we allow for discontinuous controls, the meaning of the solution of the above differential equation has to be clarified, in fact, it means that the following integral equation holds:

$$\forall\, t \in [0,T], \quad \dot{\gamma}(t) = \int_0^t f(\gamma(\tau)) + \sum_{i=1}^{m} u_i(\tau) g_i(\gamma(\tau))\, \mathrm{d}\tau.$$

To emphasize the dependence of the trajectories on the input u, we denote by γ_u the trajectory of Σ generated by $u(\cdot)$ and starting from $\gamma(0) = x_0$. We denote by $\mathcal{U}$ the class of admissible inputs, that is the set of U-valued functions defined on $[0,T]$ such that the associated trajectory γ_u exists for all $t \in [0,T]$, and we endow it with the L^2-norm topology. The domain $\mathcal{U}$ is not the whole $L^2([0,T],U)$ because of the explosion phenomena. For instance, consider the control-affine system $\dot{x} = x^2 + u$, then γ_u is not defined on $[0,T]$ for $u = 1$ if $T \geq \frac{\pi}{2}$. We now formulate the problem of motion planning (MPP).

Problem (Motion Planning Problem)**.** Consider a control-affine system Σ. For any fixed point $x^\star \in \mathcal{X}$, find a control $u^\star \in \mathcal{U}$ such that $\gamma_{u^\star}(T) = x^\star$.

The MPP can be decomposed into two subproblems, which are both difficult to solve in their full generality. First is the accessibility/controllability one, i.e. describing the subset of $\mathcal{X}$ that can be reached from x_0 in time T using the controls of $\mathcal{U}$. This question may be answered by studying the structure of the Lie algebra of the system (we recall some results of this theory in the following paragraph). Second is the problem of actually finding a control u^* that realises the transfer from x_0 to $x^\star$. Many different algorithmic methods have been developed throughout the years, some of them are presented in section 2. Those two problems are related and they need to be solved in order. Indeed, given a control system we need first to establish the existence of a control $u^\star$ that steers x_0 to $x^\star$, and next we have to implement a numerical method to compute actually $u^\star$. The algorithm that we present in the next chapter is devoted to solving the second subproblem of the MPP only.

Remark (Extension to control-nonlinear systems). Notice that the MPP can also be considered for general control-nonlinear systems $\Xi \;:\; \dot{x} = F(x,w)$, where F is a smooth map and w is the control. But for three reasons it is convenient to consider control-affine systems only. Firstly, as we discussed in the first part of the thesis, any control-nonlinear system Ξ can be prolonged to a control-affine system by augmenting its state space with the controls w and introducing new controls $u_i = \dot{w}_i$, which gives

$$\Xi^p : \begin{cases} \dot{x} &= F(x, w) \\ \dot{w} &= u \end{cases} .$$

The second reason is that using control-affine systems allows to endow the set of controls with the L^2-norm topology and therefore we benefit from its hilbertian structure. And, thirdly, for many applications (for instance, in robotics) the control systems that we consider are control-affine, or even control-linear (see below) systems. ◆

We call Σ control-linear or *driftless* if $f \equiv 0$; those systems are given by

$$\Lambda : \dot{x} = \sum_{i=1}^{m} u_i g_i(x).$$

They can be interpreted as kinematical systems under nonholonomic constraints (i.e. constraints on the velocities), those constraints are given by differential one-forms ω_i (smooth section of $T^*\mathcal{X}$), for $i = m+1, \ldots, n$, spanning the codistribution $\operatorname{ann}(\operatorname{span}\{g_1, \ldots, g_m\})$. From the point of view of the applications, control-linear systems model kinematical systems whose state represents the position and we control the velocities directly. As opposed to systems whose state represents positions and velocities, and where the controls are forces or torques.

Review of Accessibility and Controllability results. In this paragraph, we present some results about the controllability of control-linear systems Λ, the interested reader will find more details in [BJR98; Jur96]. Consider a control-linear system of the form

$$\Lambda : \dot{x} = \sum_{i=1}^{m} u_i g_i(x), \quad x \in \mathcal{X}, \quad u \in U.$$

We define the reachable set from $x_0 \in \mathcal{X}$ in time $T > 0$ by $R_T(x_0) = \{\gamma_u(T),\ u \in \mathcal{U}\}$; this set describes the set of points that can be reached from x_0 in time T using the controls of $\mathcal{U}$. We call the system Λ controllable (from x_0) if $\cup_{T \geq 0} R_T(x_0) = \mathcal{X}$, and controllable in time T if $R_T(x_0) = \mathcal{X}$.

Definition 7.1 (Lie Algebra Rank Condition)**.** We say that the vector fields $g_1, \ldots, g_m$ satisfy the *Lie Algebra Rank Condition* (shortly, LARC) if for all $x \in \mathcal{X}$ we have

$$\operatorname{Lie}(g_1, \ldots, g_m)(x) = T_x\mathcal{X},$$

where the left hand side denotes the Lie algebra generated by the vector fields $g_1, \ldots, g_m$ evaluated at $x \in \mathcal{X}$.

We can now recall the celebrated Chow-Rashevsky theorem which gives a necessary condition for controllability ([Cho40; Ras38]).

Theorem 7.2 (Rashevsky-Chow)**.** *If $\mathcal{X}$ is connected and if the vector fields g_i of Λ satisfy the LARC at every $x \in \mathcal{X}$, then any two points x_0 and x_1 of $\mathcal{X}$ can be joined by a trajectory of Λ, i.e. $\exists\, u$ such that $\gamma_u(0) = x_0$ and $\gamma_u(T) = x_1$.*

For a proof see [BJR98]. As a corollary, when the LARC holds then Λ is *controllable* from any point x_0. The converse of Rashevsky-Chow theorem, that is if Λ is controllable then the g_i's and their iterated Lie brackets span the tangent space at every point of $\mathcal{X}$ is true if $\mathcal{X}$ and the vector fields g_i are analytic and false in the C^∞ case (see [Sus73]).

Endpoint mapping. In this paragraph, we present the main properties of the input-output map, equivalently the *endpoint* map, associated with a control-affine system. All results are quoted from [Tré00] and we only add some comments. We consider a control affine system Σ of the form

$$\Sigma \,:\, \dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x), \quad x \in \mathcal{X}, \quad u \in U.$$

Recall that x_0 and $T > 0$ are fixed, we denote by $\mathcal{U} \subset L^2([0,T],U)$ the set of controls such that the trajectories γ_u of Σ, starting form x_0, are well defined on $[0,T]$. We endow $\mathcal{U}$ with the L^2-norm topology.

Definition 7.3 (Endpoint map)**.** The endpoint mapping is

$$\begin{aligned} E \,:\, \mathcal{U} &\longrightarrow \mathcal{X} \\ u &\longmapsto \gamma_u(T), \end{aligned}$$

i.e. it associates to each control the terminal point of its corresponding trajectory.

Observe that the problem of controllability is equivalent to the description of the image of the endpoint mapping. In particular, if a system is controllable then E is surjective. Let u and $(u_n)_{n\in\mathbb{N}}$ be elements of $L^2([0,T],U)$, we denote by $u_n \rightharpoonup u$ the weak convergence of the sequence u_n to u in L^2. We consider the linearised system along a trajectory γ_u:

$$\dot{y}_v(t) = A_u(t) y_v + B_u(t) v, \quad v \in L^2, \tag{7.1}$$

where $A_u = \frac{\partial f}{\partial x}(\gamma_u) + \sum_{i=1}^m u_i \frac{\partial g_i}{\partial x}(\gamma_u)$, and $B_u = (g_1(\gamma_u), \ldots, g_m(\gamma_u))$. Let M_u be the $(n \times n)$-matrix solution of the equation $\dot{M}_u = A_u M_u$, $M_u(0) = \mathrm{Id}_n$. We have the following properties of the endpoint mapping.

Proposition 7.1 (Properties of the endpoint map)**.** *The endpoint mapping E of a smooth control-affine system Σ satisfies the following properties:*

(i) The domain $\mathcal{U}$ of E is open in $L^2([0,T],U)$;

(ii) If $u_n \rightharpoonup u$, then γ_{u_n} is well defined on $[0,T]$ for a sufficiently large n and, additionally, $\gamma_{u_n} \to \gamma_u$ uniformly on $[0,T]$;

(iii) The endpoint map is L^2-Fréchet differentiable, and we have

$$\forall\, v \in \mathcal{U}, \quad \mathrm{d}E(u)\cdot v = y_v(T) = M_u(T) \int_0^T M_u(\tau)^{-1} B_u(\tau) v(\tau)\, \mathrm{d}\tau; \tag{7.2}$$

(iv) If $u_n \rightharpoonup u$, then $\mathrm{d}E(u_n) \to \mathrm{d}E(u)$.

In statement *(iii)*, we claim that E is L^2-differentiable, usually, one proves that E is differentiable on L^∞, see [Tré05].

Remark (Singular controls). A control u_s (or its associated trajectory $\gamma_{u_s}(t)$) is said to be *singular* on $[0,T]$ if u_s is a singular point of the endpoint map, i.e. the Fréchet differential of E is not surjective at u_s. Otherwise we say that u is regular. ◆

A control u in the interior of $\mathcal{U}$ is singular if and only if the linearised system (7.1) along the trajectory γ_u is not controllable (see [BC03]). We will denote by $\mathcal{S} \subset \mathcal{U}$ the set of singular controls. Moreover, we have the following characterisation of singular controls.

Proposition 7.2 (Hamiltonian characterisation of singular controls)**.** *Let u_s be a singular control of Σ on $[0,T]$, and let γ_{u_s} be the associated singular trajectory. Then there exists an absolutely continuous map $p : [0,T] \to \mathbb{R}^n \backslash \{0\}$, called adjoint vector, such that the following equations are fulfilled for almost every $t \in [0,T]$:*

$$\begin{aligned}
&\dot{\gamma}_{u_s}(t) = \frac{\partial H}{\partial p}(\gamma_{u_s}(t), p(t), u_s(t)),\\
&\dot{p}(t) = -\frac{\partial H}{\partial x}(\gamma_{u_s}(t), p(t), u_s(t)),\\
&\frac{\partial H}{\partial u}(\gamma_{u_s}(t), p(t), u_s(t)) = 0,
\end{aligned}$$

where $H(x,p,u) = \langle p, f(x) + \sum_{i=1}^m u_i g_i(x)\rangle$ is the hamiltonian, and $\langle \cdot, \cdot \rangle$ is the duality between one-forms and vector fields on $\mathcal{X}$.

That proposition provides a geometric interpretation of the adjoint vector $p(t)$. If u_s is singular on $[0,T]$, then it is also singular on $[0,t]$, for all $t \leq T$, moreover, $p(t)$ is orthogonal to the image of $\mathrm{d}E_t(u_s)$*. In particular, $\mathrm{im}\,(\mathrm{d}E_t(u_s))$ is a subspace of $T\mathcal{X}$ of codimension greater or equal than 1.

Having defined the map $\mathrm{d}E(u) : \mathcal{U} \to T_{\gamma_u(T)}\mathcal{X}$, it will be convenient to determine its adjoint $\mathrm{d}E(u)^* : T^*_{\gamma_u(T)}\mathcal{X} \to \mathcal{U}$ explicitly. Using the same notations as for the linearised system (7.1), we consider $p_T \in T^*_{\gamma_u(T)}\mathcal{X}$ and we set $p(t)$ (seen as a time-varying vector) to be the solution of the following differential equation, called the adjoint equation,

$$\dot{p}(t) = -A_u^*(t)p(t),$$

with terminal condition $p(T) = p_T$. Then, we have

$$\left(\mathrm{d}E(u)^* p_T\right)(t) = B_u^*(t)p(t).$$

The map $\mathrm{d}E(u)$ is surjective if and only if $\mathrm{d}E(u)^*$ is one-to-one. Moreover, $\mathrm{d}E(u)^*$ fails to be one-to-one if and only if there exists a nontrivial solution $p(t)$ of the adjoint equation such that $\langle p(t), g_i(\gamma_u(t))\rangle$ vanishes identically for $i = 1, \ldots, m$.

*E_t is the endpoint map at time $t > 0$.

2 Review of motion planning methods

We call a motion planning method an algorithm that solves the problem of finding a control $u^\star \in \mathcal{U}$ such that the generated trajectory $\gamma_{u^\star}$ satisfies $\gamma_{u^\star}(0) = x_0$ and $\gamma_{u^\star}(T) = x^\star$, for a chosen target $x^\star \in \mathcal{X}$. Today, there is no algorithm that guarantees any control system to reach an accessible goal exactly. In this section, we present some existing approaches in the motion planning area and we try to give a taste of the advantages and drawbacks of each method. In particular, to compare those methods with each other we will use the criteria introduced by Long in [Lon10]. We will call a motion planning method a *complete procedure* if all properties of the following list are fulfilled:

(i) Generality: the method should work for any control system without any restrictions on its structure;

(ii) Globality: for every pair of points $(x_0, x^\star) \in \mathcal{X} \times \mathcal{X}$ the algorithm should produce a control that steers the system from x_0 to a point $\tilde{x}^\star \in \mathcal{X}$ arbitrary close to $x^\star$;

(iii) Proof: a mathematical proof guaranteeing item *(ii)* should be given;

(iv) Usability: The algorithm must be implementable regardless of the system:

(a) No restriction on the dimension of the system,
(b) Robust with respect to the dynamics of the system,
(c) Produce regular trajectories,
(d) Generalisable to any domain, in particular, with obstacles.

The following Table 2.1 summarises the main characteristics of the methods presented below with respect to those four criteria.

	Generality	Globality	Proof	Usability
Lafferriere-Sussmann	✗	✓	✓	✗
Murray-Sastry	✗	✓	✓	✓
Liu-Sussmann	✗	✓	✓	✗
Flatness	✗	✓	✓	✗
Optimal control	✓	✓	✗	✓
Continuation	✓	✓	✓	✓

Table 2.1: Summary of the characteristics of existing motion planning methods

Most of the presented techniques apply to control-linear systems only (and sometimes to some subclasses of control-linear systems), only the methods based on optimal control, and the continuation method can argue to apply to any control-nonlinear system. We do not go into the details of each method, the interested reader will find supplementary informations in [Lau98; Sus92], [BL05, Chapter 13], and references therein.

2.1 Lie bracket based methods

In this subsection, we present three methods based on the study of the Lie algebra associated with a control-linear system Λ. Throughout the subsection, we assume that the vector fields $g_1, \dots, g_m$ of Λ satisfy the LARC at every $x \in \mathcal{X}$. Therefore, those methods are not applicable to a general control system. The basic idea is that in Rashevsky-Chow theorem one assumes that the vector fields $g_1, \dots, g_m$ together with their iterated Lie brackets span all the directions of the tangent space. So if one can make the system follow the directions of the Lie brackets, then one can go in any direction. This is achieved as follows. Assume that $f_1, \dots, f_n$ are vector fields locally forming a basis of the tangent space. Denote by $e^{t_i f_i} x_0$ the flow of f_i passing through x_0 at $t_i = 0$. Then there is a small neighbourhood of x_0 on which the maps $(t_1, \dots, t_n) \mapsto e^{t_1 f_1 + \dots + t_n f_n} x_0$ and $(t_1, \dots, t_n) \mapsto e^{t_1 f_1} \dots e^{t_n f_n} x_0$ are two coordinate systems, called the first and the second normal coordinate system, respectively. Thanks to the Baker–Campbell–Hausdorff formula [Hal15] a relation between the two systems is given (for a sufficiently small τ) by

$$e^{\tau f_i} e^{\tau f_j} = e^{\tau f_i + \tau f_j + \frac{1}{2}\tau^2 [f_i, f_j] + \tau^2 \varepsilon(\tau)},$$

where $\varepsilon(\tau) \to 0$ when $\tau \to 0$. Hence, $e^{-\tau f_i} e^{-\tau f_j} e^{\tau f_i} e^{\tau f_j} \approx e^{\tau^2 [f_i, f_j]}$, so following the flow of f_j, then of f_i, then of $-f_j$, and finally of $-f_i$, for a small time τ, is the same as following the flow of $[f_i, f_j]$ for τ^2.

Lafferriere-Sussmann: Nilpotent approximation. We present one of the first methods used to solve the MPP. It was developed by Lafferriere and Sussmann in [LS93], and it gives an exact solution for nilpotent systems and an approximative solution in the general case. A control-linear system is called nilpotent as soon as the iterative Lie brackets of the vector fields $g_1, \dots, g_m$ vanish starting from some given length. Suppose that $\mathcal{B} = \{g_1, \dots, g_m, g_{m+1}, \dots, g_r\}$, r might be greater than n, constitute a P. Hall basis around a point x_0, see [Lau93]; notice that due to the LARC we have $\mathcal{B}(x) = T_x \mathcal{X}$ for any $x \in \mathcal{X}$. We define the extended system as

$$\Lambda_e \,:\, \dot{x} = \sum_{i=1}^{m} u_i g_i + \sum_{i=m+1}^{r} v_i g_i.$$

Then for any path γ satisfying $\gamma(0) = x_0$ and $\gamma(T) = x^\star$ we can express $\dot{\gamma}$ in terms of $\mathcal{B}$. The resulting coefficients define a control (u, v) that steers the extended system along γ. The second step consists in reducing the control (u, v) to a control u of the original system. Since the system is nilpotent, every Lie bracket can be exactly expressed by a finite combination of the vector fields $g_1, \dots, g_m$ (see [Lon10] for a detailed construction). For a general control-linear system, Lafferriere and Sussmann propose to use the above method by considering only Lie brackets up to some length k. This method produces a trajectory ending as close to the goal as wanted (when $k \to \infty$).

The method of Lafferriere-Sussmann is very interesting as it applies to all control-linear systems and it produces an exact solution in the nilpotent case and an approximate solution in the general case. Nevertheless, there are some computational drawbacks. The method requires some change of coordinates that, when numerically performed, induce instability and quadrature errors. Moreover the P. Hall basis is defined locally only, thus making the method global appears to be a difficult task in practice.

Murray-Sastry method for chained systems. Roughly at the same time as Lafferriere and Sussmann, Murray and Sastry explored in [MS93] the use of sinusoidal inputs with integer frequencies to solve the MPP for the class of chained systems. We consider a chained system of the form

$$\dot{x} = u_1 \begin{pmatrix} 1 \\ 0 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The idea is to control the above form component by component. The key point is to ensure that if the component x_i is moving during $[0, T]$, then no other component x_k, for $k < i$, changes (i.e. $x_k(0) = x_k(T)$). That strategy is possible with sinusoidal inputs, the authors proposed to use $u_1 = a\sin(\omega_1 t)$ and $u_2 = b\cos(\omega_2 t)$, where a and b are parameters depending on the initial and final state of the system, and ω_1 and ω_2 are suitably chosen integer frequencies. See [MLS17, Proposition 8.3] for a detailed proof of the efficiency of the proposed method. See also [TMS95] for an application of that method to the n-trailer system. Murray-Sastry's method is interesting because it is simple to implement, but its is applicable to a small class of systems only. The method has been generalised in [CJL13] to be applicable to a more general form of chained systems, nevertheless that generalisation has not been implemented yet.

Liu-Sussmann: Oscillatory controls. We finish this subsection by presenting a method introduced by Liu and Sussmann in [SL91], details can be found in [Liu97] and a worked out example is in [SL93]. That method combines ideas from the two above ones. Using the extension Λ_e (see above) of a control-linear system Λ given by a P. Hall basis, it is easy to solve the MPP. Indeed, every smooth curve joining x_0 and $x^\star$ is a trajectory of Λ_e. Based on this observation, Liu in [Liu97] shows that we can explicitly construct a sequence of controls (u^j) such that the corresponding generated trajectories γ_{u^j} of Λ converge uniformly to γ (when $j \to \infty$). That result allows to solve the MPP in an approximate way, indeed it is enough to find a curve γ such that $\gamma(0) = x_0$ and $\gamma(T) = x^\star$ and then to construct the sequence u^j. Choosing $j^\star$ large enough, the control $u^{j^\star}$ steers the system arbitrarily close to $x^\star$ (see [Sus91, Algorithm 4]).

The main advantage of this method is that it applies to any control-linear system, it depends only on the structure of the Lie algebra generated by the vector fields $g_1, \ldots, g_m$. However, this approach seems to be difficult to implement in practice. Indeed, we need to take sinusoidal control with frequencies tending to infinity (even for nilpotent systems) in order to obtain the convergence toward the goal.

2.2 Other methods

The three methods that we present in this subsection are applicable to any control-nonlinear system $\Xi : \dot{x} = F(x, u)$, where F is a smooth map from $\mathcal{X} \times U \to \mathcal{X}$. There exists many other approaches to solve the motion planning problem, such as probabilistic, multi-level, optimisation, etc, see [Lau98] for further details.

Differential flatness. Differential flatness has been introduced in the early nineties by Fliess, Lévine, Martin, and Rouchon [Fli+95; Rou+93]. The main idea lies in an explicit parametrisation of the trajectories of control-nonlinear systems. A flat system is a control system for which there exists m of independent variables y, that are functions of the state, of the controls, and of their successive time-derivative, such that the controls and the trajectories of the system can be expressed as functions of y and a finite number of their time-derivatives. Precisely,

Definition 7.4 (Flat system)**.** We say that a control-nonlinear system $\Xi : \dot{x} = F(x,u)$ is *flat* if there exits integers $r \geq -1$ and $l \geq 0$, and smooth functions $h : \mathcal{X} \times U \times \mathbb{R}^{rm} \to \mathbb{R}^m$, $\psi : \mathbb{R}^{m(l+2)} \to U$, and $\Psi : \mathbb{R}^{m(l+1)} \to \mathcal{X}$ such that if we set

$$y = h(x, u, \dot{u}, \dots, u^{(r)}),$$

called a *flat output* of Ξ, then the inputs and the trajectories of Ξ are given by

$$u = \psi(y, \dot{y}, \dots, y^{(l+1)}), \quad \text{and} \quad x = \Psi(y, \dot{y}, \dots, y^{(l)})$$

This definition shows that if a system is flat, then its dynamics is that of a control-linear system (in an extended state space). Thanks to that property, it is easy to do the motion planning for a flat system (for which we explicitly know a flat output). By definition we have,

$$x_0 = \Psi(y(0), \dot{y}(0), \dots, y^{(l)}(0)), \quad \text{and} \quad x^\star = \Psi(y(T), \dot{y}(T), \dots, y^{(l)}(T)). \tag{7.3}$$

Thus it is enough to find a curve $t \mapsto y(t)$ in the space of the flat outputs satisfying the conditions of (7.3). This can be done by polynomial interpolation for instance. Finally, a solution of the MPP is given by $u^\star(t) = \psi(y(t), \dot{y}(t), \dots, y^{(l+1)}(t))$.

The main drawback of this method is that there does not exist a general characterisation of flat systems. Moreover, even if a system is flat, then there does not exist an explicit method to find a flat output. Furthermore, flatness is not a generic property of control systems. For instance, a control-linear system with two controls is flat if and only if it is static feedback equivalent to the chained form [MR94; MMR01; LR12] (in the latter all flat outputs are described). A generic control-linear system with $m = 2$ controls is flat on an open and dense subset of $\mathcal{X}$, if $\dim \mathcal{X} = 3$ or $\dim \mathcal{X} = 4$. On the other hand, if $\dim \mathcal{X} \geq 5$, then flat systems form a very tiny subset of all control-linear systems with two controls. Nevertheless, a big number of systems met in practice are flat and, when it is the case, flatness methods appear to be very useful and powerful, see [MMR03].

Optimal control. Optimal control is one of the most important topic in the study of control-nonlinear systems. Indeed, the idea of associating a cost to each admissible trajectory and trying to find the trajectory minimising that cost seems to solve two problems at once. On one hand, if we can solve optimal controls problems, then we get an admissible trajectory and thus solve the MPP, and on the other hand we obtain a trajectory minimising some criterion. Formally, optimal control problems for $\dot{x} = F(x,u)$ are formulated as

$$u^\star = \underset{u \in \mathcal{U},\ \gamma_u(T) = x^\star}{\arg\min} \int_0^T F^0(\tau, \gamma_u(\tau), u(\tau))\, \mathrm{d}\tau + g(\gamma_u(T), u(T)).$$

Under some regularity assumptions, one can show that an optimal control exists (see e.g. [HSV95] for a presentation of some general results). Usually, there are two types of numerical methods used to solve optimal control problems, namely, direct and indirect methods. Direct methods consist in discretising the cost function, the state space, and the control space, this procedure reduces the optimal control problem to a problem of nonlinear optimisation under constraints. In particular, among direct methods we find an approach via dynamic programming which yields the resolution of Hamilton-Jacobi-Bellman (HJB) equation. Indirect methods aim at solving with a shooting method a problem obtained from the application of the Pontryagin Maximum Principle (see [Tré05, chapter 9] and [Tré12] for a detailed presentation of those methods with a lot of practical examples).

The main difficulties of optimal control approaches are: in general we will get local minima, in the case of direct methods this is due to the discretisation while for indirect methods it is due to the Maximum Principle which is a necessary condition only. Secondly, those methods are costly as soon as the dimension of the state increases, moreover, indirect methods may have a small convergence domain (they rely on Newton's algorithm). Notice that the approach via HJB's equation guarantees that we find a global minimum, but that approach is numerically expensive. Nevertheless, using optimal control to solve the MPP is probably the best method that we have at our disposal.

Homotopy continuation method. The objective of the continuation method is to solve the MPP step by step from a simpler one by a parameter deformation. That approach requires first to characterise the singularities of the endpoint mapping and next to prove the global existence of a highly nonlinear differential equation on the space of controls, the path lifting equation. We leave a more detailed presentation of the continuation method and of its characteristics to the next chapter as our proposed algorithm is a modification of it. We stress that this method has the advantage that it applies to any kind of control system and that it is easy to implement, but a proof of its globality amounts to show that the path lifting equation has a global solution which is hard in general. Theoretical results have been obtained by Sussmann and Chitour in [Sus93; Chi06] for control-linear systems under some quite restrictive controllability assumptions.

Chapter 8

A regularised continuation method

In this chapter, we discuss the application of the continuation method to the motion planning problem. First, we review its main properties in the case of smooth control-linear systems, then we propose a regularisation, based on the Tikhonov regularisation in Moore-Penrose pseudo-inverse theory, of that method and we analyse its properties. In particular, we study the convergence of a solution of our regularised approach to a solution of the motion planning problem. Finally, we propose a numerical implementation of our regularised method and we illustrate its potential through several numerical examples.

1 Introduction

Throughout the chapter, we consider a control-affine system of the form

$$\Sigma \,:\, \dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x), \quad x \in \mathbb{R}^n, \quad u_i \in L^2([0,T], \mathbb{R}).$$

The driftless, or control-linear, systems (i.e. Σ with $f \equiv 0$) will be denoted by Λ. We suppose that an initial state $x_0 \in \mathbb{R}^n$ is fixed and we set the final time $T > 0$. The set $\mathcal{U} \subset L^2([0,T], \mathbb{R}^m)$ consists in the controls u for which the associated trajectory $x_u(t)$ is well-defined for every $t \in [0,T]$. We denote by

$$\begin{aligned} E \,:\, \mathcal{U} &\longrightarrow \mathbb{R}^n \\ u &\longmapsto x_u(T), \end{aligned}$$

the endpoint mapping. Its properties are summarised in Proposition 7.1 of Chapter 7. Throughout the chapter, we assume that Σ (or Λ) is completely controllable in time T, i.e. the map E is surjective.

Presentation of the continuation method. We now present the continuation method in a general setting, we will show later how to apply this method to the motion planning problem. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, satisfying $\dim \mathcal{H}_2 < \infty$, and let $E : \mathcal{H}_1 \to \mathcal{H}_2$ be a smooth surjective map. Fix $y^\star \in \mathcal{H}_2$, the purpose of the continuation method is to determine $x^\star \in \mathcal{H}_1$ such that

$$E(x^\star) = y^\star, \tag{8.1}$$

i.e. to find a preimage of $y^\star$. This method has been introduced in the context of numerically solving nonlinear equations, see e.g. [AG90; CL15], and proceeds as follows. We begin by choosing an arbitrary point $x^0 \in \mathcal{H}_1$ and we denote $y^0 = E(x^0)$, then we construct a smooth path $\pi(s)$ between y^0 and $y^\star$, that is, $\pi : [0,1] \to \mathcal{H}_2$ such that $\pi(0) = y^0$ and $\pi(1) = y^\star$. The key step in the method is the «lifting» of π into a path $\Pi : [0,1] \to \mathcal{H}_1$ satisfying

$$\forall\, s \in [0,1], \quad E(\Pi(s)) = \pi(s). \tag{8.2}$$

In particular, notice that $\Pi(0) = x^0$ satisfies that relation at $s = 0$. If the procedure of finding $\Pi(s)$ can be carried out to $s = 1$ then $x^\star = \Pi(1)$ is a solution of equation (8.1). Since Π is defined implicitly, one usually proceeds by differentiating (8.2) with respect to s, which yields

$$\mathrm{d}E(\Pi(s))\frac{\mathrm{d}\Pi}{\mathrm{d}s}(s) = \frac{\mathrm{d}\pi}{\mathrm{d}s}(s), \quad s \in [0,1], \quad \Pi(0) = x^0.$$

If this equation admits a global solution on $[0,1]$, then $x^\star = \Pi(1)$ is a solution of our original problem. Suppose that $\mathrm{d}E(\Pi(s))$ has full rank for all $s \in [0,1]$, then $\mathrm{d}E(\Pi(s))$ admits a right inverse $P(\Pi(s))$, for instance one can take the Moore-Penrose pseudo-inverse defined by

$$P(\Pi(s)) = \mathrm{d}E(\Pi(s))^* \left(\mathrm{d}E(\Pi(s))\mathrm{d}E(\Pi(s))^*\right)^{-1},$$

where * stands for the adjoint operator. Choosing $\frac{\mathrm{d}\Pi}{\mathrm{d}s}(s)$ as $P(\Pi(s))\frac{\mathrm{d}\pi}{\mathrm{d}s}(s)$, leads to the following ordinary differential equation on $\mathcal{H}_1$:

$$\frac{\mathrm{d}\Pi}{\mathrm{d}s}(s) = P(\Pi(s))\frac{\mathrm{d}\pi}{\mathrm{d}s}(s), \quad s \in [0,1], \quad \Pi(0) = x^0. \tag{8.3}$$

We call equation (8.3) the *path lifting equation* (PLE). If one can find a solution of (8.3), with the initial condition $\Pi(0) = u^0$, which is defined on the whole interval $[0,1]$, then $\Pi(1)$ is a solution of our original equation (8.1). Equation (8.3) is a *Ważewski equation* see [Waż47] and [Ole98]. The two issues of the continuation method are the following:

(i) **non-degeneracy**: it is necessary that $\mathrm{d}E(\Pi(s)$ has full rank at all $s \in [0,1]$, that is $\pi(s) \notin E(\mathcal{S})$, where $\mathcal{S}$ is the singular set of E;

(ii) **non-explosion**: equation (8.3) must have a global solution on $[0,1]$ with $E(\Pi(0)) = \pi(0)$.

If one of the above conditions fails to hold, then the continuation method may not work as illustrated by the example below.

Example. From [Sus92, section 7]. Take $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}$ and the application $E : \mathcal{H}_1 \to \mathcal{H}_2$ given by $E(x) = x^3 - 3x$. From $E'(x) = 3x^2 - 3$ we obtain the singular set $\mathcal{S} = \{-1, 1\}$ and $E(\mathcal{S}) = \{-2, 2\}$. Then the path $\pi(s) = -3 + 6s$ (connecting $x^0 = -3$ and $x^\star = 3$) crosses $E(\mathcal{S})$ and thus cannot be globally lifted. This can be seen on its Ważewski equation $\Pi'(s) = \frac{2}{\Pi(s)^2 - 1}$, with $\Pi(0)$ the real solution of $x^3 - 3x = -3$, which clearly admits a maximal solution $\Pi(s)$ defined on $s \in [0, 5/6[$ and $\Pi(s)$ approaches -1 when $s \to \frac{5}{6}$. ●

Moore-Penrose Pseudo-Inverse. We recall the main properties of the Moore-Penrose Pseudo-Inverse (MPPI) which plays an important role in the continuation method (both for the classical and our regularised version). Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, with $\dim \mathcal{H}_2 < \infty$, we denote $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ their respective scalar product. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a continuous linear mapping and let $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ be its adjoint, i.e. for all $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$ we have $(Ah_1, h_2)_2 = (h_1, A^*h_2)_1$. We set $M = AA^*$, which is a linear map from $\mathcal{H}_2 \to \mathcal{H}_2$ satisfying $M \geq 0$. Clearly, if A is onto then M is invertible (in particular, $M > 0$). In that case, we define

$$\begin{aligned} A^+ : \mathcal{H}_2 &\longrightarrow \mathcal{H}_1 \\ h_2 &\longmapsto A^+h_2 = A^*M^{-1}h_2. \end{aligned}$$

It follows that we have the identity $AA^+h_2 = h_2$ for all $h_2 \in \mathcal{H}_2$, so A^+ is a right inverse of A. Hence, A^+ fulfils the axioms of the Moore-Penrose pseudo-inverse:

$$AA^+A = A, \quad A^+AA^+ = A^+, \quad AA^+ \text{ and } A^+A \text{ are self-adjoint}, \tag{8.4}$$

so, by definition, the operator A^+ is the MPPI of A (it is unique since A has closed range). If $M = AA^*$ is not invertible, there is an alternative procedure to obtain A^+ (i.e. the unique operator satisfying (8.4)). The matrix $AA^* + \lambda \mathrm{Id}_{\mathcal{H}_2}$ is invertible for λ small enough. Then, we claim that

$$A^+ = \lim_{\lambda \to 0} A^* \left(AA^* + \lambda \mathrm{Id}_{\mathcal{H}_2}\right)^{-1}. \tag{8.5}$$

See [BH12, Lemma 4.2 and Theorem 4.3] for a proof. We now review some of the main properties of the MPPI, for more details see [BT21] for a historical introduction, [BH12] for the case of matrices, and [Beu65a; Beu65b] for a general setting.

We set the operator $P = \mathrm{Id}_{\mathcal{H}_2} - AA^+$ from $\mathcal{H}_2$ to $\mathcal{H}_2$, then P is an orthogonal projector (i.e. $P^* = P$ and $P^2 = P$) satisfying

$$\ker P = \operatorname{im} A, \quad \text{and} \quad \operatorname{im} P = \ker A^+, \tag{8.6}$$

in particular, if A is onto then $P = 0$. Consider now the (overdetermined) linear equation $Ah_1 = h_2$, that we treat by a least square problem, i.e. we search the set of vectors $h_1 \in \mathcal{H}_1$ minimizing the norm $\|Ah_1 - h_2\|^2$. Then, the set of vectors of $\mathcal{H}_1$ for which the previous least square problem assumes a minimum is given by the set

$$A^+h_2 + \ker A.$$

Consequently, A^+h_2 is the vector of smallest length such that $Ah_1 = h_2$. In particular, it shows that the operator A^+ does not depend on the inner product of $\mathcal{H}_2$. Finally, the solutions of the regularised least square problem

$$\min_{h_1 \in \mathcal{H}_1} \|Ah_1 - h_2\|^2 + \lambda \|h_1\|^2, \quad \lambda > 0$$

are given by $h_1 = A^* \left(AA^* + \lambda \mathrm{Id}_{\mathcal{H}_2}\right)^{-1} h_2$, it is a straightforward computation from the so-called normal equation $(A^*A + \lambda \mathrm{Id}_{\mathcal{H}_2})h_1 = A^*h_2$.

Continuation method adapted to the MPP. We now explain how the idea of the continuation method applies to the motion planning problem. In the context of motion planning, the continuation method has been introduced by Sussmann [Sus92; Sus93] and developed by Chitour [CS98; Chi06]. See also [ACL10] for a practical implementation of that method applied to the rolling body problem.

In the context of motion planning, the role of the surjective map above is played by the endpoint mapping $E : \mathcal{U} \to \mathbb{R}^n$, that assigns to each control u the terminal point $x_u(T)$ of the associated generated trajectory (starting from x_0 and in time $T > 0$). Therefore, the first step of the continuation requires the determination of the singular set $\mathcal{S}$ of E (i.e. the controls u such that $\operatorname{rk} \mathrm{d}E(u) < n$) and the choice of a path $\pi \in \mathbb{R}^n$ which avoids the image $E(\mathcal{S})$ of the singular set. Next, we pick a control u^0 and we choose a path $\pi : [0,1] \to \mathbb{R}^n$ such that $\pi(0) = E(u^0)$, $\pi(1) = x^\star$, and $\pi(s) \notin E(\mathcal{S})$. The second step consists in lifting $\pi(s)$ into a path $\Pi(s) \in \mathcal{U}$ verifying $E(\Pi(s)) = \pi(s)$ for all $s \in [0,1]$. The lift $\Pi(s)$ should verify equation (8.3) which is now an ordinary differential equation on $\mathcal{U}$:

$$\mathrm{d}E(\Pi(s))\frac{\mathrm{d}\Pi}{\mathrm{d}s}(s) = \frac{\mathrm{d}\pi}{\mathrm{d}s}(s), \quad s \in [0,1], \quad \Pi(0) = u^0. \tag{8.7}$$

The existence of the global solution of (8.7) must be established. See figure 8.1 for a visual summary of the method.

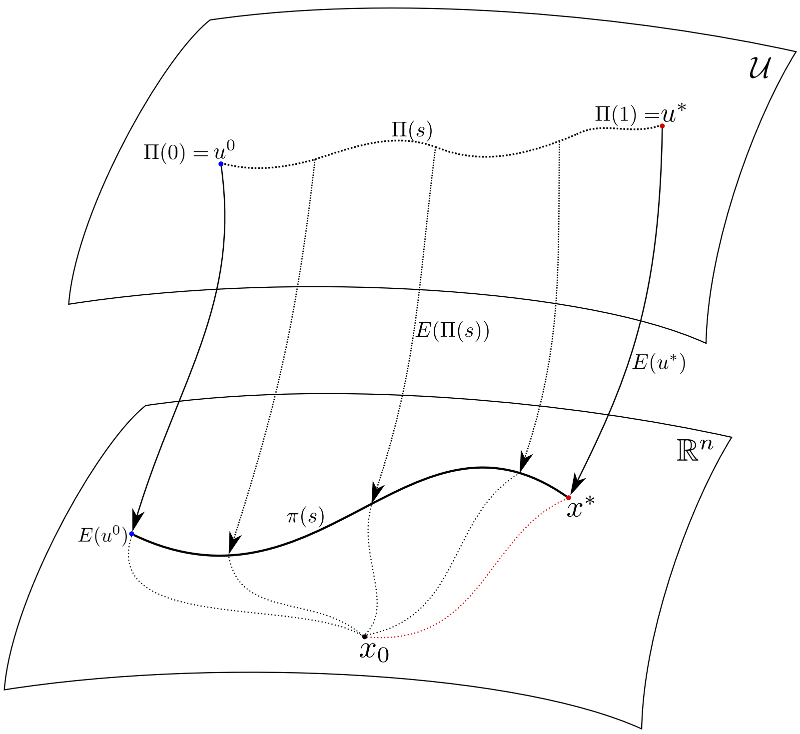


Figure 8.1: representation of the continuation method applied to motion planning. The top, resp. bottom, shape represents the space of controls $\mathcal{U}$, resp. the state space $\mathbb{R}^n$. The arrows from top to bottom represents the application of the endpoint map E, the thick solid black line is the path $\pi(s)$ between $x^0 = E(u^0)$ and $x^\star$, and the dotted black line on $\mathcal{U}$ is the constructed path $\Pi(s)$.

As long as $\pi(s) \notin E(\mathcal{S})$, the differential $\mathrm{d}E(\Pi(s))$ is right invertible, consequently, to solve equation (8.7), we use its Moore-Penrose pseudo-inverse $P(\Pi(s)) : T_{E(\Pi(s))}\mathbb{R}^n \to$

$\mathcal{U}$ defined by:

$$\forall\, u \in \mathcal{U}\backslash\mathcal{S}, \quad P(u) = \mathrm{d}E(u)^* G(u)^{-1},$$

where the map $G(u) : T_{E(u)}\mathbb{R}^n \to T_{E(u)}\mathbb{R}^n$ defined by $G(u) = \mathrm{d}E(u)\mathrm{d}E(u)^*$ is the controllability Gramian. And we choose $\Pi'(s)$ equal to

$$\Pi'(s) = P(\Pi(s))\pi'(s), \quad s \in [0,1], \quad \Pi(0) = u^0, \tag{8.8}$$

and in this way (8.7) is satisfied. Recall that equation (8.8) is called the *path lifting equation* (PLE). We now assume that we work with control-linear systems Λ. Assuming that Λ is completely controllable implies that the singularities of E are exactly the controls giving rise to the abnormal extremals. The existence of nontrivial abnormal extremals (see [Mon93; LS94]) leads to hard problems such as the precise determination of the singular set. We review the different properties of the continuation method due to [Chi06]. Recall that we assume that the domain $\mathcal{U}$ of the endpoint map is exactly the set of controls such that $x_u(T)$ exists (therefore the assumption of Chitour [Chi06] that Λ must be a CC-tempered system is fulfilled). First, we have

Proposition 8.1 (Local existence and uniqueness)**.** *Let $\pi : [0,1] \to \mathbb{R}^n\backslash\overline{E(\mathcal{S})}$ be a C^1 path. Then, for all pairs $(\bar{s}, \bar{u}) \in [0,1] \times \mathcal{U}\backslash\mathcal{S}$, we have local existence and uniqueness of the maximal solution $\Pi(s)$ of (8.8) with initial condition $\Pi(\bar{s}) = \bar{u}$.*

See [Chi06, Proposition 2] for a proof. Before turning to global existence of the PLE, we have the following proposition that puts forward a finite dimensional version of the PLE on any compact interval J of the domain of existence I of the maximal solution Π of the PLE. More precisely, we consider a strictly increasing sequence $\mathcal{U}_j$ of finite dimensional subspaces of $\mathcal{U}$ such that $\cup_{j\geq 0}\mathcal{U}_j$ is dense in $\mathcal{U}$. Let $\Pi : I \to \mathcal{U}$ be the maximal solution of PLE with initial solution $u^0 \in \mathcal{U}_0$. Set E_j the endpoint map associated to $\mathcal{U}_j$ The proposition states that for any compact subinterval $J = [0, s_0] \subset I$, the PLE defined by E_j and associated with π has a global solution on J, for j large enough. In particular, if the PLE has a global solution, i.e. $I = [0,1]$, then J can be taken as $[0,1]$, and thus, the proposition shows that the PLE will have a global solution in a finite-dimensional subspace of $\mathcal{U}$ thus justifying to use Galerkin procedures to solve numerically the PLE.

Notations. Let $\mathcal{U}_j$ be a closed linear subspace of $\mathcal{U}$, we set pr_j the orthogonal projection onto $\mathcal{U}_j$. The image of E_j is equal to that of $E\,\mathrm{pr}_j : \mathcal{U} \to \mathbb{R}^n$. For $u \in \mathcal{U}$, let $\mathrm{d}E_j(u) = \mathrm{d}E(u)\,\mathrm{pr}_j$ and $\mathrm{d}E_j(u)^*$ be its adjoint (relative to the scalar product on $\mathcal{U}_j$), and $G_j(u) = \mathrm{d}E_j(u)\mathrm{d}E_j(u)^*$ be the controllability Gramian restricted to $\mathcal{U}_j$. If $\mathrm{d}E_j(u)$ is onto, then we set $P_j(u) = \mathrm{d}E_j(u)^* G_j(u)^{-1}$ the Moore-Penrose pseudo-inverse of $\mathrm{d}E_j(u)$.

Proposition 8.2 (Finite dimensional reduction of the PLE)**.** *Let $\pi : [0,1] \to \mathbb{R}^n\backslash\overline{E(\mathcal{S})}$ be a C^1 path and $(\mathcal{U}_j)$ be a strictly increasing sequence of finite-dimensional subspaces of $\mathcal{U}$ such that $\cup_{j\geq 0}\mathcal{U}_j$ is dense in $\mathcal{U}$. Assume that $u^0 \in \mathcal{U}_0$, and let I be the interval of existence of the maximal solution Π of the PLE starting at u^0. Then, for every compact subinterval $J = [0, s_0] \subset I$, there exists j so that for every $j \geq$ j, the equation*

$$\Pi'(s) = P_j(\Pi(s))\pi'(s), \quad s \in J, \quad \Pi(0) = u^0,$$

admits a global solution in J.

See [Chi06, Theorem 1] for a proof. Next, the following proposition relates the regularity of the controls $\Pi(s)$ to the one of the initial condition u^0. Independently of the space $\mathcal{U}$ of controls, the solution Π of a PLE is as regular as its initial condition u^0. We define the Sobolev spaces H^k, for some integer $k \geq 0$, by

$$H^k([0,T],\mathbb{R}^m) = \left\{u \in L^2([0,T],\mathbb{R}^m),\ \forall\,\alpha \text{ s.t. } |\alpha| \leq k,\ D^\alpha u \in L^2([0,T],\mathbb{R}^m)\right\},$$

where α is a multi-index, and D^α is a partial derivative of u (in the sense of distributions). Those are Hilbert spaces and, in particular, we have $H^0 = L^2$.

Proposition 8.3 (Regularity of the controls $\Pi(s)$)**.** *Let $\pi : [0,1] \to \mathbb{R}^n \backslash \overline{E(\mathcal{S})}$ be a smooth path.*

(i) Assume that $\mathcal{U} = L^2([0,T],\mathbb{R}^m)$ and that $u^0 \in H^k([0,T],\mathbb{R}^m)$ for some integer $k \geq 0$. Let I be the interval of existence of the maximal solution Π of the PLE starting at u^0. Then, $\forall\, s \in I$, $\Pi(s)$ can be written as $u^0 + M(s)$, with $M(s) \in H^{k+1}([0,T],\mathbb{R}^m)$.

(ii) The same conclusion holds, when we replace H^k with C^k.

See [Chi06, Theorem 2] for a proof. In particular, if u^0 is smooth and if the PLE has a global solution, then the control $u^\star = \Pi(1)$ solving the *MPP* is also smooth.

Results establishing global existence of the PLE, have been proven by Sussmann and then extended by Chitour under the additional requirement that Λ satisfies the strong bracket generating condition.

Definition 8.1 (SBGC)**.** We say that Λ satisfies the *strong bracket generating condition* (SBGC) if

$$\forall\,\theta \in \mathbb{R}^m\backslash\{0\},\ \forall\,x \in \mathbb{R}^n,\ \theta \cdot g := \sum_{i=1}^{m} \theta_i g_i,$$
$$\text{the vectors } g_1(x), \ldots, g_m(x), \left[\theta \cdot g, g_1\right](x), \ldots, \left[\theta \cdot g, g_m\right](x) \text{ span } T_x\mathbb{R}^n.$$

Under the SBGC, there do not exist any nontrivial abnormal extremals, in other words, it turns out that $\mathcal{S}$ reduces to $\{0\}$, see [Str86]. The following proposition asserts that if the SBGC holds not only the PLE is well-posed for every $s \in [0,1]$, but also its solution $\Pi(s)$ exists for all $s \in [0,1]$.

Proposition 8.4 (Global existence for the PLE)**.** *Consider a control-linear system satisfying the LARC and the SBGC, and for which $\mathcal{U} = H^k([0,T],\mathbb{R}^m)$, for some $k \geq 0$. Then, for every C^1-path π such that $x_0 \notin \pi([0,1])$, the PLE (8.8) admits a global solution on $[0,1]$.*

That proposition is a combination of [Chi06, Theorems 3 and 4].

2 Regularisation of the continuation method

We now present a regularised version of the continuation method. We present our approach in the context of the MPP, nevertheless we emphasize that this method can be carried out on general applications of the continuation method. The major difficulty of the classical continuation method arises from the singularities of the endpoint map, our regularisation overcomes this difficulty. Indeed, the proposed method simultaneously tackles the non-degeneracy and the non-explosion conditions of the PLE, its cost is, however, the introduction of a family of problems whose solutions may not converge to a solution of the original problem. We present our results for control-linear systems Λ satisfying the LARC (see Definition 7.1), some of our results can be extended to other class of completely controllable control systems without much extra work.

Recall that the path lifting equation is given, when $\mathrm{d}E(\Pi(s))$ is onto, in terms of the Moore-Penrose pseudo-inverse $P(\Pi(s))$ of $\mathrm{d}E(\Pi(s))$. Equation (8.5) shows that the MPPI can be approximated by a family of linear operators, thus it suggests to introduce the following equation parametrised by $\lambda > 0$

$$\frac{\mathrm{d}\Pi_\lambda}{\mathrm{d}s}(s) = \mathrm{d}E(\Pi_\lambda(s))^* \left(\mathrm{d}E(\Pi_\lambda(s))\mathrm{d}E(\Pi_\lambda(s)) + \lambda \mathrm{Id}_{\mathbb{R}^n}\right)^{-1} \frac{\mathrm{d}\pi}{\mathrm{d}s}(s), \tag{8.9}$$

with initial condition $\Pi_\lambda(0) = u^0$. We call this equation the *regularised path lifting equation* (R-PLE, shortly). To simplify notations, we will denote $\frac{\mathrm{d}\Pi_\lambda}{\mathrm{d}s}$ by Π'_λ.

Remark (Interpretation of the R-PLE as a regularised least square problem). Using basic properties of the MPPI, we deduce that the R-PLE can equivalently be formulated as

$$\Pi'_\lambda(s) = \arg\min_{v\in\mathcal{U}} \|\mathrm{d}E(\Pi_\lambda(s))v - \pi'(s)\|^2_{\mathbb{R}^n} + \lambda \|v\|^2_{\mathcal{U}}. \tag{8.10}$$

This formulation is interesting as it shows that we can imagine many other types of regularisation, via replacing $\lambda \|v\|^2$ by any functional $\psi(v)$ with suitable properties. ◆

The following proposition shows that the R-PLE is well posed and that it takes care of the issues of degeneracy and of explosion of the classical PLE. Our result does not require that the path $\pi(s)$ avoids the singularities of E. Hence, our regularised method avoids the main difficulty of the classical continuation method.

Proposition 8.5 (Existence and global solution of the R-PLE)**.** *Assume $\pi(s)$ is a C^1-curve on $\mathbb{R}^n$. Then, for any $\lambda > 0$ and any $u^0 \in \mathcal{U}$ the regularised path lifting equation, with initial condition $\Pi_\lambda(0) = u^0$, admits a global solution on $[0, 1]$.*

The same conclusion applies to control-nonlinear systems under the additional assumption that $0 \in \mathcal{U}$, i.e. that the trajectory associated with the zero control is well defined on $[0, T]$; for control-affine systems Σ, it means that the trajectory of the dynamical system $\dot{x}(t) = f(x(t))$ exists for all $t \in [0, T]$.

Proof. As outlined in the remark above, the R-PLE can be rewritten as

$$\Pi'_\lambda(s) = \arg\min_{v\in\mathcal{U}} \|\mathrm{d}E(\Pi_\lambda(s))v - \pi'(s)\|^2_{\mathbb{R}^n} + \lambda\|v\|^2_{\mathcal{U}}.$$

For all $v \in \mathcal{U}$ we have

$$\begin{aligned}\|\mathrm{d}E(\Pi_\lambda(s))\Pi'_\lambda(s) - \pi'(s)\|^2_{\mathbb{R}^n} + \lambda\|\Pi'_\lambda(s)\|^2_{\mathcal{U}} &\le \|\mathrm{d}E(\Pi_\lambda(s))v - \pi'(s)\|^2_{\mathbb{R}^n} + \lambda\|v\|^2_{\mathcal{U}}\\ \lambda\|\Pi'_\lambda(s)\|^2_{\mathcal{U}} &\le \|\mathrm{d}E(\Pi_\lambda(s))v - \pi'(s)\|^2_{\mathbb{R}^n} + \lambda\|v\|^2_{\mathcal{U}},\end{aligned}$$

In particular, for $v = 0$ (which obviously is in $\mathcal{U}$ for a control-linear system) we get $\lambda\|\Pi'_\lambda(s)\|^2_{\mathcal{U}} \le \|\pi'(s)\|^2_{\mathbb{R}^n}$, concluding the proof. ∎

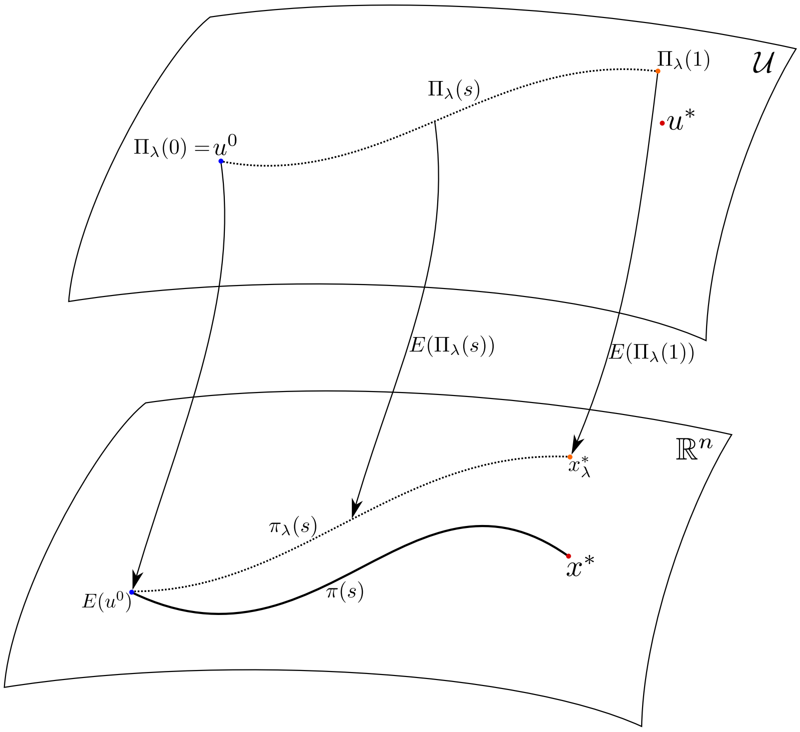


Figure 8.2: Representation of the regularised continuation method applied to motion planning. In addition to figure 8.1, the dotted black line on $\mathbb{R}^n$ represents the image $\pi_\lambda(s)$ of the regularised solution $\Pi_\lambda(s)$ by the endpoint map E.

The process of the regularised continuation method is summarised in figure 8.2. The natural question that we will answer next is the convergence of the global solution of the R-PLE to a solution of the PLE as $\lambda \to 0$. Given the solution $\Pi_\lambda(s)$ of the R-PLE, we construct, on the state space $\mathbb{R}^n$, the path

$$\pi_\lambda(s) := E(\Pi_\lambda(s)), \quad \forall\, s \in [0,1],$$

which is different from $\pi(s)$ except for $s = 0$ (see figure 8.2). We give a necessary condition for the path $\pi(s)$ assuring that $\lim_{\lambda\to 0} \pi_\lambda(1) = x^\star$. To this end, we define the error due to the regularisation by

$$e_\lambda(s) = \pi(s) - \pi_\lambda(s).$$

Observe that for all λ, we have $e_\lambda(0) = 0$, therefore if $e'_\lambda(s)$ is close to zero (or even vanishes for almost every s), then the error at $s = 1$ should be small. We differentiate e_λ with respect to s and we get

$$\begin{aligned} e'_\lambda(s) &= \pi'(s) - \frac{\mathrm{d}}{\mathrm{d}s}\left(E(\Pi_\lambda(s))\right), \\ &= \pi'(s) - \mathrm{d}E(\Pi_\lambda(s))\Pi'_\lambda(s), \\ &= \left[\mathrm{Id} - \mathrm{d}E(\Pi_\lambda(s))\mathrm{d}E(\Pi_\lambda(s))^* \left(\mathrm{d}E(\Pi_\lambda(s))\mathrm{d}E(\Pi_\lambda(s))^* + \lambda\mathrm{Id}\right)^{-1}\right]\pi'(s). \end{aligned}$$

For a general linear map A, we have $\lim_{\lambda\to 0} AA^*(AA^* + \lambda Id)^{-1} = AA^+$ and recall from (8.6) that the operator $P = \mathrm{Id} - AA^+$ is an orthogonal projector such that $\ker P = \operatorname{im} A$ and $\operatorname{im} P = \ker A^+$. With our analysis, we conclude:

Proposition 8.6. *Suppose that for all $s \in [0,1]$, there exists $\Pi_0(s) \in \mathcal{U}$ such that $\Pi_\lambda(s) \xrightarrow{\lambda \to 0} \Pi_0(s)$. If*

$$\pi'(s) \in \operatorname{im} \mathrm{d}E(\Pi_0(s)), \quad \textit{for almost every } s \in [0,1], \tag{8.11}$$

then the global solution $\Pi_\lambda(s)$ of the regularised path lifting equation converges to a solution of the motion planning problem when λ goes to zero.

Observe that for the classical continuation method, we need that $\pi(s) \notin E(\mathcal{S})$ in order for the PLE to be well-posed. In our regularised setting, condition (8.11) only restricts the derivative $\pi'(s)$ of the path. That condition means that the path π can pass through the image of the singular set if it crosses it transversally. Therefore, we have replaced a condition formulated on the state space by a condition formulated on the tangent space.

Proof. Using Proposition 7.1 *(iv)* we obtain that $\mathrm{d}E(\Pi_\lambda) \to \mathrm{d}E(\Pi_0)$, so the analysis performed above the proposition holds and we get $\lim_{\lambda\to 0} e'_\lambda(s) = 0$. Hence, $\lim_{\lambda\to 0} e_\lambda(1) = 0$ implying that $\lim_{\lambda\to 0} \pi_\lambda(1) = \pi(1) = x^\star$. ■

Existence of $\Pi_0(s)$ as above does not follow from classical optimisation arguments because, in the minimisation problem (8.10), the linear operator $\mathrm{d}E(\Pi_\lambda(s))$ depends non trivially on λ. Nevertheless, one can prove that if $\Pi_\lambda(s)$ weakly converges to $\Pi_0(s)$ for all s, then $\Pi'_0(s)$ is a solution of the following limit minimisation problem

$$\min_{v \in \mathcal{U}} \left\| \mathrm{d}E(\Pi_0(s))v - \pi'(s) \right\|^2,$$

see e.g. [Cla21]. Observe that, under condition (8.11), the above minimum is zero and we have $\mathrm{d}E(\Pi_0(s))\Pi'_0(s) = \pi'(s)$ implying that $\Pi_0(s)$ is a solution of the equation $E(\Pi(s)) = \pi(s)$ for all $s \in [0,1]$.

To summarise, a necessary condition for the convergence of the regularised continuation method is $\pi'(s) \in \operatorname{im} \mathrm{d}E(\Pi_0(s))$, which implies that we have to construct a path $\pi(s)$ such that $\pi'(s)$ is not orthogonal to $\operatorname{im} \mathrm{d}E(\Pi_0(s))$. If, for every s, the control $\Pi_0(s)$ is not singular then $\mathrm{d}E(\Pi_0(s))$ has full rank and condition (8.11) is automatically fulfilled. However, if for some $\bar{s}$ the control $\Pi_0(\bar{s})$ is singular, then the image of $\mathrm{d}E(\Pi_0(\bar{s}))$ is a subspace of $\mathbb{R}^n$ with codimension at least 1. In that case, condition (8.11) is non-trivial, and $\pi(s)$ has to be chosen accordingly. This transversality condition is the crucial point of the method. In the future, we want to tailor this condition and to find strategies to construct such paths. Moreover, as we already mentioned, it is difficult to establish the convergence of $\Pi_\lambda(s)$; in the future we will try to obtain a convergence result for the regularised path lifting equation.

3 Discretisation and description of the algorithm

In this section, we present some technical algorithmic aspects on the implementation of our method. Consider a control-nonlinear system of the form

$$\Xi : \dot{x} = F(x,u), \quad x \in \mathcal{X}, \quad u(\cdot) \in U.$$

Suppose that an initial state $x_0 \in \mathcal{X}$ and a final time T are fixed. Let E be the endpoint mapping associated to Ξ and assume that $x^\star$ is reachable from x_0.

Algorithm 1 Regularised continuation method for the MPP

Choose $\lambda > 0$;
Choose an arbitrary control $u^0 \in \mathcal{U}$ and set $x^0 = E(u^0)$;
Define a curve $\pi : [0,1] \to \mathcal{X}$ such that $\pi(0) = x^0$ and $\pi(1) = x^\star$;
Solve numerically the R-PLE

$$\Pi'_\lambda(s) = \mathrm{d}E(\Pi_\lambda(s))\left(\mathrm{d}E(\Pi_\lambda(s))\mathrm{d}E(\Pi_\lambda(s))^* + \lambda \mathrm{Id}\right)^{-1}\pi'(s), \quad \Pi_\lambda(0) = u^0; \tag{8.12}$$

Set $u^\star_\lambda = \Pi_\lambda(1)$;

Observe that the main difficulty of Algorithm 1 is to solve the R-PLE, which is an ordinary differential equation defined on the control space $\mathcal{U}$. In the following paragraphs, we develop some key points for solving (8.12).

Discretizing the control space. We start by approximating the control space $\mathcal{U}$ which is an open set in an infinite dimensional vector space. We divide the interval $[0,T]$ into H parts $\{t_1,\ldots,t_H\}$, and we approximate the control space $\mathcal{U}$ by the mH-dimensional subspace $\hat{\mathcal{U}}$ of piecewise linear functions. Then, for $1 \le i \le m$, a control u_i is approximated by $\hat{u}_i$, the linear interpolation of $(u_{i,1},\ldots,u_{i,H})$, where $u_{i,k} = u_i(t_k)$, i.e. we have (see figure 8.3)

$$\hat{u}_i(t) = u_{i,k} + (t - t_k)\frac{u_{i,k+1} - u_{i,k}}{t_{k+1} - t_k}, \quad t \in [t_k, t_{k+1}].$$

The regularised path lifting equation (8.12) tells us how we have to modify the piecewise approximation $\hat{u}^0$ in order to obtain an approximate control steering our system from its initial state to a point near the target state $x^\star$.

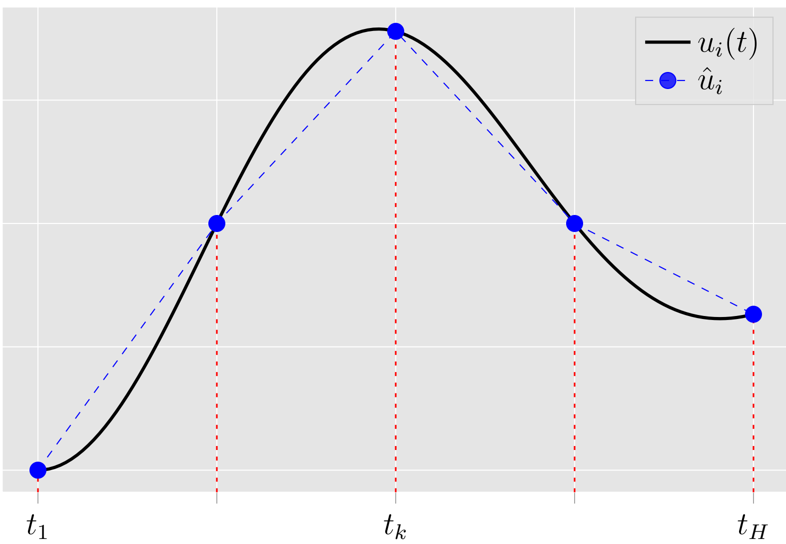


Figure 8.3: Discretisation of the control space. Black thick line displays a control $u_i(t)$, the blue dots and the dashed line represent its finite dimensional discretisation and approximation.

For all $\hat{u}$ and $\hat{v}$ in $\hat{\mathcal{U}}$, the L^2 scalar product is discretised as follows:

$$\begin{aligned}\langle \hat{u}, \hat{v}\rangle_{\hat{\mathcal{U}}} &= \sum_{i=1}^{m} \int_0^T \hat{u}_i(t)\hat{v}_i(t)\,\mathrm{d}t = \sum_{i=1}^{m}\sum_{k=1}^{H-1} \int_{t_k}^{t_{k+1}} \hat{u}_i(t)\hat{v}_i(t)\,\mathrm{d}t,\\ &= \sum_{i=1}^{m}\sum_{k=1}^{H-1} \Delta t_k \left(u_{i,k}v_{i,k} + \frac{1}{2}u_{i,k}\Delta v_{i,k} + \frac{1}{2}v_{i,k}\Delta u_{i,k} + \frac{1}{3}\Delta u_{i,k}\Delta v_{i,k}\right), \end{aligned} \tag{8.13}$$

where $\Delta t_k = t_{k+1} - t_k$, $\Delta u_{i,k} = u_{i,k+1} - u_{i,k}$, and $\Delta v_{i,k} = v_{i,k+1} - v_{i,k}$. We have chosen the space of piecewise linear controls because they are easy to implement, but we plan to test our method on other Galerkin approximations of the control space.

Computing the differential of the endpoint map. Let $\hat{u} \in \hat{\mathcal{U}}$ be a piecewise linear control and consider $x_{\hat{u}}$ the trajectory (starting at x_0 and defined on $[0, T]$) generated by $\hat{u}$. We describe how to compute a matrix representation of $\mathrm{d}E(\hat{u}) : \hat{\mathcal{U}} \to T_{E(\hat{u})}\mathcal{X}$ and of $\mathrm{d}E(\hat{u})^* : T^*_{E(\hat{u})}\mathcal{X} \to \hat{\mathcal{U}}$. The space $\hat{\mathcal{U}}$ is an open set in a mH-dimensional vector space, therefore it is possible to compute a matrix representation of the linear map $\mathrm{d}E(\hat{u})$ using a basis $\{e_i\}$ of $\hat{\mathcal{U}}$, this would require mH numerical integrations of the linearised system around the trajectory $x_{\hat{u}}$. But since $\dim T_{E(\hat{u})}\mathcal{X} = n \ll mH$, it is much more efficient to compute a matrix representation of $\mathrm{d}E(\hat{u})^*$ and then to use the properties of the adjoint operator to get a matrix representation of $\mathrm{d}E(\hat{u})$. Precisely, we perform the linearisation of the control-nonlinear system Ξ along the trajectory $x_{\hat{u}}$ by setting

$$A_{\hat{u}}(t) = \frac{\partial F}{\partial x}(x_{\hat{u}}(t), \hat{u}(t)) \quad \text{and} \quad B_{\hat{u}} = \frac{\partial F}{\partial u}(x_{\hat{u}}(t), \hat{u}(t)).$$

Let $p_{\hat{u}} : [0, T] \to T^*\mathcal{X}$ be the field of covectors along $x_{\hat{u}}$ satisfying the adjoint equation along $x_{\hat{u}}$ with terminal condition $p_T \in T^*_{E(\hat{u})}\mathcal{X}$, i.e.

$$\dot{p}_{\hat{u}}(t) = -A^*_{\hat{u}}(t)p_{z,\hat{u}}(t), \quad p_{z,\hat{u}}(T) = p_T. \tag{8.14}$$

Then, $\mathrm{d}E(\hat{u})^*p_T \in \hat{\mathcal{U}}$ is defined by

$$\left(\mathrm{d}E(\hat{u})^*p_T\right)(t) = B^*_{\hat{u}}(t)p_{\hat{u}}(t). \tag{8.15}$$

Finally, to obtain an $(mH\times n)$-matrix representation of $\mathrm{d}E(\hat{u})^*$ it suffices to integrate n times equation (8.14) with terminal conditions $p_T \in \{z_1, \dots, z_n\}$ a basis of $T^*_{E(\hat{u})}\mathcal{X}$ and to evaluate the right hand side of (8.15) at the discrete points $\{t_1, \dots, t_H\}$.

Now, to deduce an $(n \times mH)$-matrix representation of $\mathrm{d}E(\hat{u})$ from the one of $\mathrm{d}E(\hat{u})^*$, we proceed as follows. By definition, we have

$$(\mathrm{d}E(\hat{u})\hat{v}, z) = (\hat{v}, \mathrm{d}E(\hat{u})^*z)_{\hat{\mathcal{U}}}, \quad \forall\, \hat{v} \in \hat{\mathcal{U}}, \quad \forall\, z \in T^*_{E(\hat{u})}\mathcal{X}. \tag{8.16}$$

Choosing $\hat{v}$ in a basis of $\hat{\mathcal{U}}$ and z in $\{z_1, \dots, z_n\}$ we can explicitly compute the coefficients of $\mathrm{d}E(\hat{u})$. For instance, suppose that the number of controls is $m = 1$, let ϕ_k, for $1 \le k \le H$, be the basis of $\hat{\mathcal{U}}$ defined by $\phi_k(t_j) = \delta_k^j$, and let $z_i = e_i$ be the canonical basis of $\mathbb{R}^n$. Then denote E and E^* the matrix representation of $\mathrm{d}E(\hat{u})$ and $\mathrm{d}E(\hat{u})^*$, respectively. Using (8.16) with $\hat{v} = \phi_j$ and $z = z_i$ we obtain

$$\mathsf{E}_{i,j} = \sum_{k=1}^{H} \mathsf{E}^*_{k,i}\,(\phi_j, \phi_k)_{\hat{\mathcal{U}}} = \sum_{\substack{k=j-1\\ 1\le k\le H}}^{j+1} \mathsf{E}^*_{k,i}\,(\phi_j, \phi_k)_{\hat{\mathcal{U}}},$$

from which we deduce for all $1 \leq i \leq n$:

$$\mathsf{E}_{i,1} = \frac{\Delta t_1}{3}\left(\mathsf{E}^*_{1,i} + \frac{1}{2}\mathsf{E}^*_{2,i}\right), \qquad \mathsf{E}_{i,H} = \frac{\Delta t_{H-1}}{3}\left(\mathsf{E}^*_{H,i} + \frac{1}{2}\mathsf{E}^*_{H-1,i}\right),$$
$$\mathsf{E}_{i,j} = \frac{\Delta t_{j-1}}{3}\left(\mathsf{E}^*_{j,i} + \frac{1}{2}\mathsf{E}^*_{j-1,i}\right) + \frac{\Delta t_j}{3}\left(\mathsf{E}^*_{j,i} + \frac{1}{2}\mathsf{E}^*_{j+1,i}\right), \quad \forall\, 1 < j < H.$$

4 Numerical experiments

In this section, we present the results of several numerical applications of our regularised continuation method. Our algorithms have been implemented on MATLAB R2021A for a prototype and are freely available, we plan to implement our method with a more advanced programming language in view of having better computational performances.

The following parameters are set for all experiments. We fix the terminal time $T = 1$, we discretise the time interval $[0, 1]$ with 257 equally spaced points, the interval $[0, 1]$, for the variable s, of the regularised path lifting equation is discretised with 513 equally spaced points, and the space of controls is approximated by continuous piecewise linear functions on 129 equally spaced points. Finally, we linearly vary λ from 10^{-3} to 10^{-6} with 65 points. Note that the computational time of Algorithm 1 with λ fixed is on average around one minute (using a 1.9 GHz Intel Core i7).

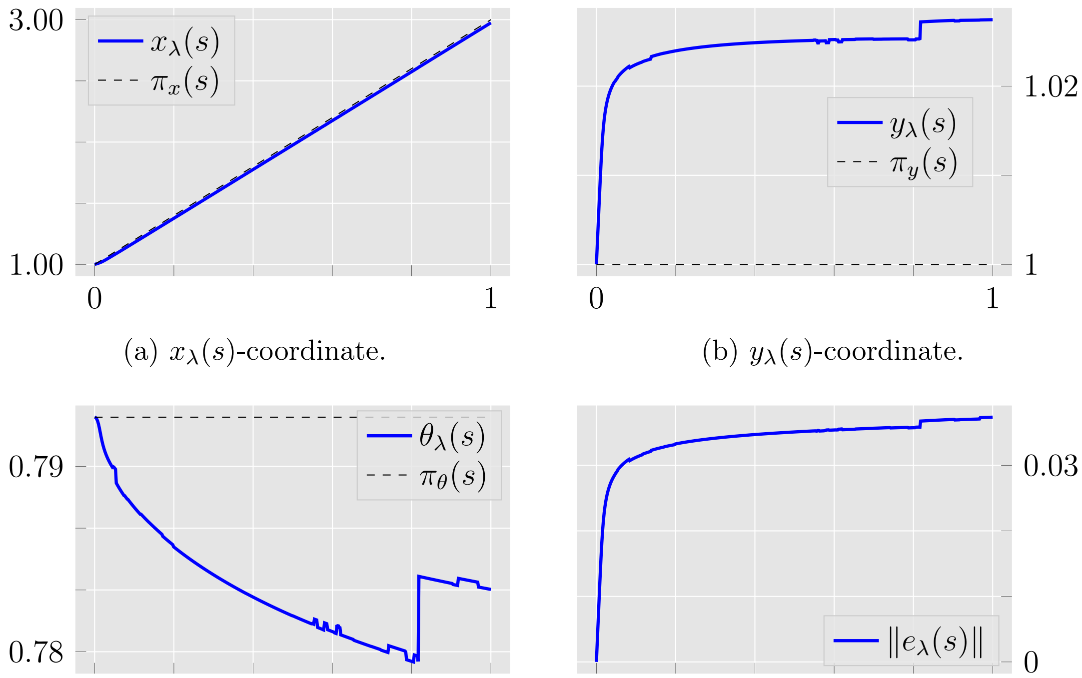
Exp. 1: Monocycle. For the first experiment, we deal with the so-called monocycle system given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2.$$

The state $(x, y, \theta)^t$ of the system represents the position and the orientation of a car on a plane. The control $u = (u_1, u_2)$ is exerted on the linear and angular velocity. This example is trivial in the sense that it does not possess any non trivial singularities; so the classical continuation method is applicable. We use this example to validate our implementation and to present different metrics used to measure the quality of our results. We use our regularised continuation method to look for a control generating a trajectory steering the system from

$$x_0 = \begin{pmatrix} 1 \\ 1 \\ \frac{\pi}{4} \end{pmatrix} \quad \text{to} \quad x^\star = \begin{pmatrix} 3 \\ 1 \\ \frac{\pi}{4} \end{pmatrix}.$$

The first control u^0 is chosen to be 0 and for the path $\pi(s)$ we choose a straight line from $x^0 = E(u^0)$ to $x^\star$. Notice that $\pi(s)$ is not an admissible trajectory of the system. First, figure 8.4 shows the results of our implementation of Algorithm 1 for $\lambda = 10^{-3}$, and second, figure 8.5 illustrates the convergence of the regularised solution when λ tends to zero.



(a) $x_\lambda(s)$-coordinate.

(b) $y_\lambda(s)$-coordinate.

(c) $\theta_\lambda(s)$-coordinate.

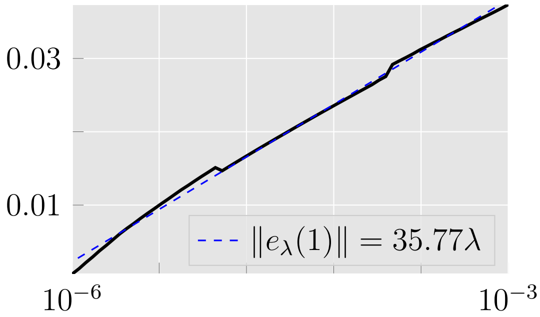
(d) Error-norm evolution.

Figure 8.4: Monocycle experiment: results of the regularised continuation method for $\lambda = 10^{-3}$. Figures 8.4(a) to 8.4(c) show the evolution for $s \in [0,1]$ of the coordinates $(x_\lambda, y_\lambda, \theta_\lambda)$ of $E(\Pi_\lambda(s))$, solid line, and the reference path $\pi(s)$, dashed line. Figure 8.4(d) shows the evolution of the error norm $\|\pi(s) - \pi_\lambda(s)\|$ for $s \in [0,1]$.

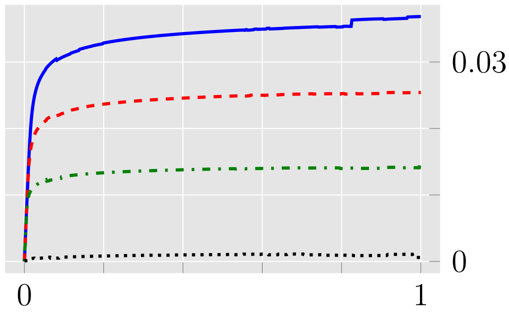
We observe that figure 8.4 shows that the regularised path lifting equation is, indeed, well-posed on $[0,1]$ and admits a global solution, illustrating Proposition 8.5. In particular, Figures 8.4(a) to 8.4(c), display the coordinates $(\pi_x, \pi_y, \pi_\theta)$ of the path $\pi(s)$ (dashed line) that join x^0 and $x^\star$, and the solid blue line shows the evolution of the coordinates of $E(\Pi_\lambda(s))$. We see that the procedure follows accurately the path $\pi(s)$ for the x coordinate, and that $\pi_\lambda(s)$ differs slightly from the reference path $\pi(s)$ for the y and θ coordinates (error of magnitude 0.01); this behaviour is expected because of the regularising parameter $\lambda = 10^{-3} > 0$.

Next, figure 8.4(d) displays the norm of the difference $\pi(s) - \pi_\lambda(s)$, i.e. the difference between the reference path π and the actually constructed one π_λ. By definition, we have $e_\lambda(0) = 0$ and we observe that $\|e_\lambda(s)\|$ is strictly increasing with s. At the end of the procedure, however, the norm of the error is not very high: the terminal point $\pi_\lambda(1)$ is in a ball of radius 0.03 around the target configuration $x^\star$. This suggests the following improvement of the method: one could use a «warm-restart» of the method, that is stopping the procedure at $s = 0.2$ and use the result of this truncated calculation as a new starting point for our algorithm (we simply need to compute a new path π).

We now study the convergence of our regularised method when the regularising parameter λ tends to 0. First, figure 8.5(a) shows that the error converges to 0 as $\lambda \to 0$, moreover it shows that the controls Π_λ converge to controls $\Pi_0(s)$, and, furthermore, it shows that there is a linear relation between the norm of the error and the regularising parameter λ. Thus, this figure proves that our regularised continuation method converges to a solution of the motion planing problem when



(a) Terminal error-norm evolution.



(b) Error-norm evolution. Solid: $\lambda = 10^{-3}$, Dashed: $\lambda = 7\,10^{-4}$, Dotted-Dashed: $\lambda = 3\,10^{-4}$, Dotted: $\lambda = 10^{-6}$.

Figure 8.5: Monocycle experiment: convergence $\lambda \to 0$ of the regularised continuation method. Figure 8.5(a) shows the evolution of the terminal error-norm $\|e_\lambda(1)\|$ against λ, solid line, and the dashed line shows the best linear approximation. Figure 8.5(b) shows the evolution of the norm $\|e_\lambda(s)\|$.

λ tends to zero. Second, figure 8.5(b) illustrates Proposition 8.6, we observe that when $\lambda \to 0$, the derivative of the error becomes smaller and thus the error tends to stay near zero during the whole continuation process. Indeed, since in this case, there are no nontrivial singularities, condition (8.11) if fulfilled everywhere.

To summarise, this first experiment validates, in a simple setting, our implementation of Algorithm 1. It shows that the regularised path lifting equation is well posed on $[0, 1]$ and possesses a global solution. Finally, we show that our regularised continuation method converges to a solution of the motion planing problem when the parameter λ tends to zero.

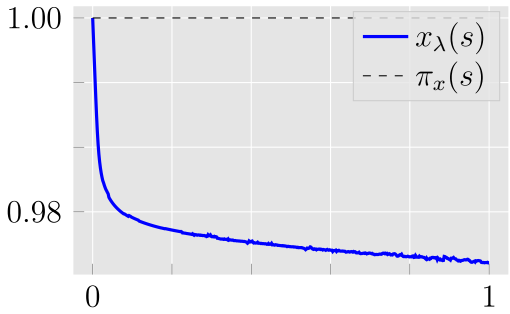
Exp. 2: Martinet case. We now turn to a more challenging example given by the Martinet system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 1 \\ 0 \\ \frac{y^2}{2} \end{pmatrix} u_2.$$

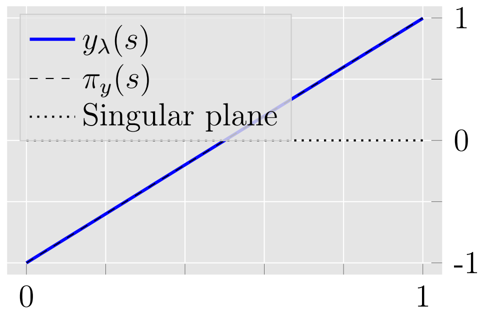
This system possesses nontrivial abnormal extremals, i.e. singularities of the end-point map E. They are the straight lines $z = z_0$ contained in the plane $\{y = 0\}$ called the Martinet plane. We test Algorithm 1 to find a control that steers the system from $x_0 = \left(\begin{smallmatrix} 1 \\ -1 \\ 1 \end{smallmatrix}\right)$ to $x^\star = \left(\begin{smallmatrix} 1 \\ 1 \\ -1 \end{smallmatrix}\right)$. We start with the initial control $u^0 = 0$, thus $x^0 = E(u_0) = x_0$, and we take $\pi(s)$ as a straight line. Therefore, observe that the path $\pi(s)$ will necessarily cross the Martinet plane.

Figure 8.6 shows the results of our implemtation of Algorithm 1 for $\lambda = 10^{-3}$. First, we observe in Figures 8.6(a) to 8.6(c) that $\pi_\lambda(s) = E(\Pi_\lambda(s))$ follows accurately the path $\pi(s)$, it deviates from that reference with an error of magnitude 0.01. Moreover observe that no special behaviour occurs when $\pi_\lambda(s)$ crosses the singular plane (at $s = 0.5$). Thus, we observe that, even in the presence of nontrivial singularities, our regularised path lifting equation is well-posed and possesses a global solution on $[0, 1]$, illustrating Proposition 8.5. Second, figure 8.6(d) shows that the error is increasing from $e_\lambda(0) = 0$ to $e_\lambda(1) \approx 0.04$, we observe that $e_\lambda(s)$

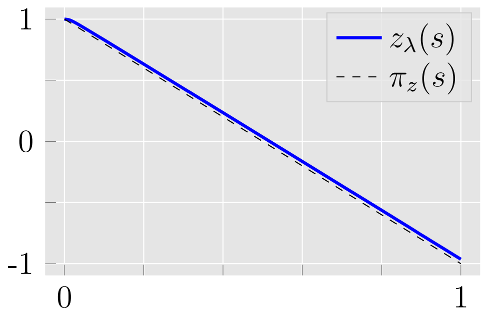
grows fast at the beginning of the path lifting equation and quickly stabilizes, which again justifies the use of a «warm-restart» procedure as described in the previous experiment.



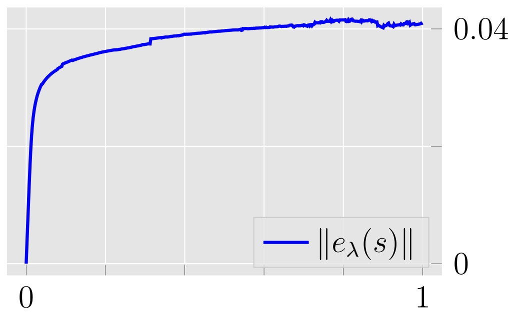
(a) $x_\lambda(s)$-coordinate.



(b) $y_\lambda(s)$-coordinate.



(c) $z_\lambda(s)$-coordinate.



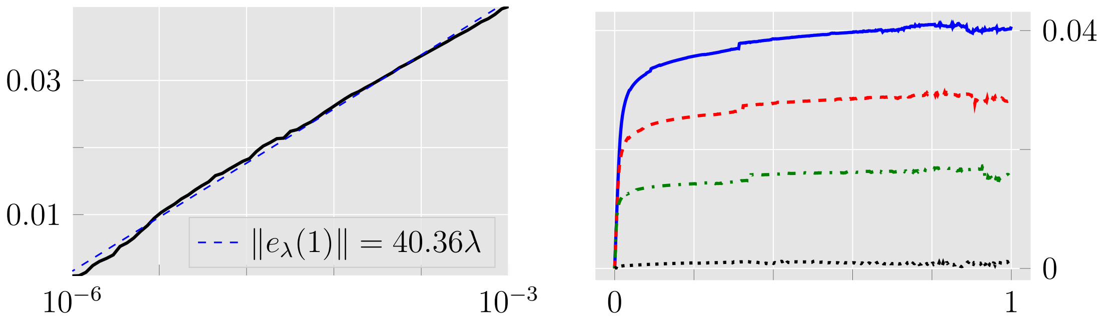
(d) Error-norm evolution.

Figure 8.6: Martinet experiment: results of the regularised continuation method for $\lambda = 10^{-3}$. Figures 8.6(a) to 8.6(c) show the evolution for $s \in [0, 1]$ of the coordinates $(x_\lambda, y_\lambda, z_\lambda)$ of $E(\Pi_\lambda(s))$, solid line, and the reference path $\pi(s)$, dashed line; moreover for the y-component we display the singular plane as a dotted line. Figure 8.6(d) shows the evolution of the error norm $\|\pi(s) - \pi_\lambda(s)\|$ for $s \in [0, 1]$.

Next, figure 8.7 shows that even when we cross singularity $\{y = 0\}$, we get convergence of the error to 0 when the regularising parameter λ tends to zero. Indeed, figure 8.7(a) shows that when λ goes to zero, then the terminal error, i.e. $\|\pi(1) - \pi_\lambda(1)\|$, also tends to zero. We also remark a linear relation between the error and λ. Next, figure 8.7(b) shows that as $\lambda \to 0$, the error $e_\lambda(s)$ gets flatter along $s \in [0, 1]$ and does not deviate a lot from 0.

We perform a second experiment with the Martinet system, which is more challenging on the crossing of the singular plane. We keep the same initial and target configurations and we *choose* a path $\pi(s)$ that follows a singular trajectory for s in an interval: first, for $s \in [0, 0.3]$, we go straight from $x^0 = (1, -1, 1)^t$ to $x^{0,1} = (1, 0, -1)^t$; second, for $s \in]0.3, 0.6]$, we go straight from $x^{0,1}$ to $x^{0,2} = (2, 0, -1)^t$, i.e. we follow a singular trajectory; and third, for $s \in]0, 1]$, we go straight from $x^{0,2}$ to $x^\star = (1, 1, -1)^t$.

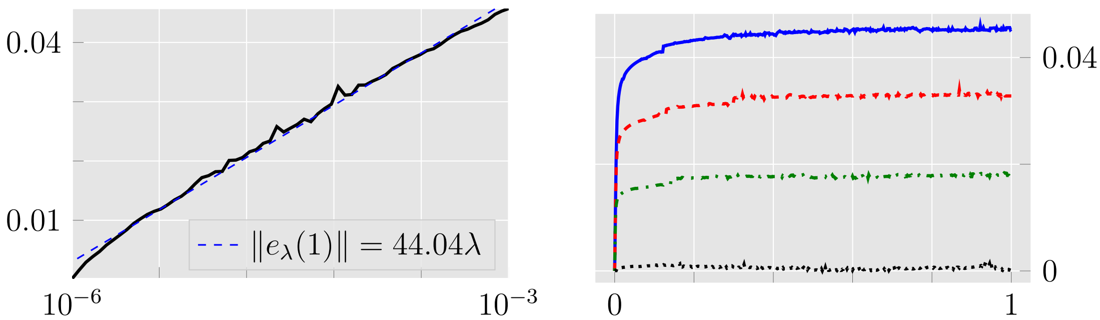
In that experiment, notice that $\pi(s)$ belongs to the singular plane $\{y = 0\}$ for s in an interval, so clearly,the classical continuation method would not be applicable here. Nevertheless, condition (8.11) is fulfilled for all s and thus we still observe convergence to zero of the terminal endpoint error $e_\lambda(1)$ and of the derivative $e'_\lambda(s)$,



(a) Terminal error-norm evolution.

(b) Error-norm evolution. Solid: $\lambda = 10^{-3}$, Dashed: $\lambda = 7\,10^{-4}$, Dotted-Dashed: $\lambda = 3\,10^{-4}$, Dotted: $\lambda = 10^{-6}$.

Figure 8.7: Martinet experiment: convergence $\lambda \to 0$ of the regularised continuation method. Figure 8.7(a) shows the evolution of the terminal error norm $\|e_\lambda(1)\|$ against λ, solid line, and the dashed line shows the best linear approximation. Figure 8.7(b) shows the evolution of the norm $\|e_\lambda(s)\|$.



(a) Terminal error-norm evolution.

(b) Error-norm evolution. Solid: $\lambda = 10^{-3}$, Dashed: $\lambda = 7\,10^{-4}$, Dotted-Dashed: $\lambda = 3\,10^{-4}$, Dotted: $\lambda = 10^{-6}$.

Figure 8.8: Martinet experiment (bis): convergence $\lambda \to 0$. Same description as figure 8.7.

see figure 8.8. Therefore, we illustrate that our method can be successfully applied to cases, where $\pi(s)$ passes through singularities for s in an interval (provided that $\pi'(s)$ is suitably chosen).

Exp. 3: Bryant system. We now turn to a more difficult academic example. The following control-linear system

$$\begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad (x, w) \in \mathbb{R}^6.$$

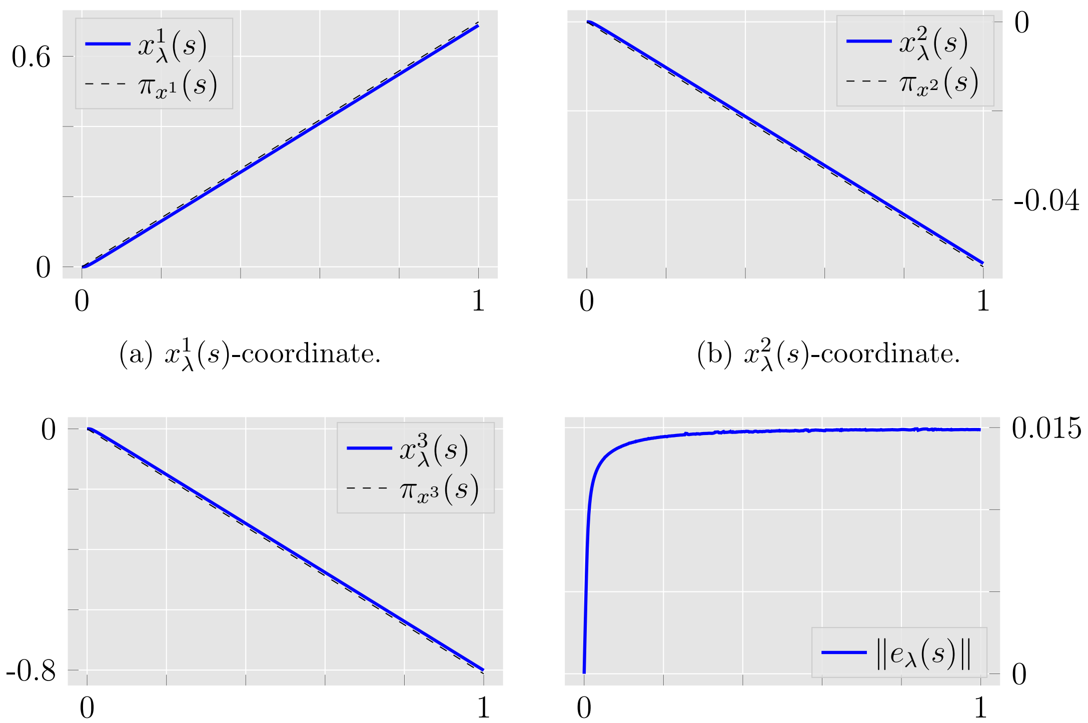
have been proposed by Bryant (unpublished) as a system possessing singular trajectories in any direction at any point. Indeed, by a straightforward calculation we get that the singular controls are $u_s(t) = v(t)p_x$, where $v(t)$ is a smooth scalar function and p_x is a constant non-zero vector of $\mathbb{R}^3$. Moreover, for a singular control u_s we have

$$E(u_s) = \begin{pmatrix} x_0 + V(T)(w_0 \wedge p_x) \\ w_0 + V(T)p_x \end{pmatrix}, \quad \text{with} \quad V(T) = \int_0^T v(\tau)\,\mathrm{d}\tau,$$

and

$$\operatorname{im} \mathrm{d}E(u_s) = \operatorname{vect}_{\mathbb{R}} \{p_x\}^{\perp} \times \mathbb{R}^3.$$

In other words, $\operatorname{im} \mathrm{d}E(u_s)$ is an hyperplane in $\mathbb{R}^6$. For our numerical experiment, we set $(x_0, w_0) = 0$ and we choose $(x^\star, w^\star)$ randomly.



(a) $x_\lambda^1(s)$-coordinate.

(b) $x_\lambda^2(s)$-coordinate.

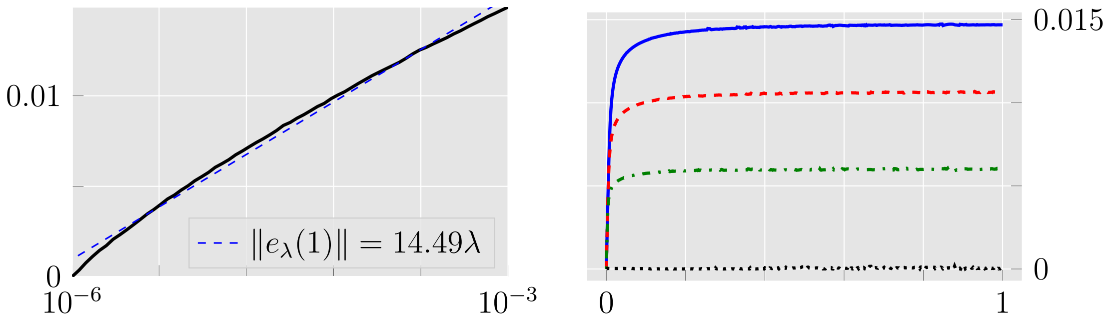
(c) $x_\lambda^3(s)$-coordinate.

(d) Error-norm evolution.

Figure 8.9: Bryant experiment: results of the regularised continuation method for $\lambda = 10^{-3}$. Figures 8.9(a) to 8.9(c) show the evolution for $s \in [0, 1]$ of the coordinate x_λ of $E(\Pi_\lambda(s))$, solid line, and the reference path $\pi(s)$, dashed line. Figure 8.9(d) shows the evolution of the error norm $\|\pi(s) - \pi_\lambda(s)\|$ for $s \in [0, 1]$.

First, as previously, we test our algorithm with $\lambda = 10^{-3}$; results are presented in figure 8.9. That figure shows that our method deals well with this delicate example. Indeed, we observe on Figures 8.9(a) to 8.9(c) that the solution $\pi_\lambda(s) = E(\Pi_\lambda(s))$ follows accurately the path $\pi(s)$ (even for $\lambda = 10^{-3}$). Moreover, figure 8.9(d) shows that the regularised path lifting equation quickly converges (with respect to s) to a solution $\Pi_\lambda(1)$, which produces a trajectory that ends near $(x^\star, w^\star)$ with an error of about 0.015. One more time, we note the potential of a «warm-restart» procedure for our algorithm.

Finally, figure 8.10 illustrates that in this delicate setting, we still get convergence of our algorithm to controls $\Pi_\lambda(s)$. In particular, figure 8.10(a) shows that



(a) Terminal error-norm evolution.

(b) Error-norm evolution. Solid: $\lambda = 10^{-3}$, Dashed: $\lambda = 7\,10^{-4}$, Dotted-Dashed: $\lambda = 3\,10^{-4}$, Dotted: $\lambda = 10^{-6}$.

Figure 8.10: Bryant experiment: convergence $\lambda \to 0$ of the regularised continuation method. Figure 8.10(a) shows the evolution of the terminal norm error $\|e_\lambda(1)\|$ against λ, solid line, and the dashed line shows the best linear approximation. Figure 8.10(b) shows the evolution of the norm $\|e_\lambda(s)\|$.

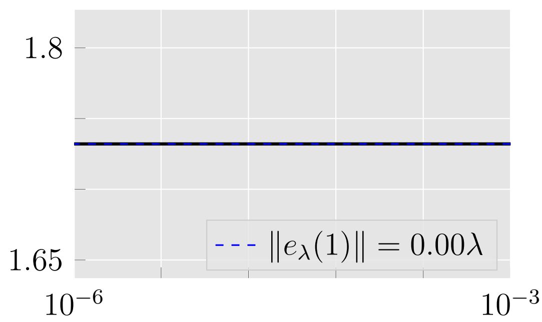
the terminal error $e_\lambda(1)$ converges (with a linear rate) to 0 when $\lambda \to 0$. Next, figure 8.10(b) shows that when $\lambda \to 0$, the error $e_\lambda(s)$ tends to be flatter and thus stays closer from 0. The fact that we obtain convergence of the error $e_\lambda(s)$ in that experiment proves that the chosen path $\pi(s)$ and the controls $\Pi_0(s)$, to which $\Pi_\lambda(s)$ have converged, satisfy condition (8.11). It is encouraging that our method works well on such system with «many» singularities, when we choose randomly the target state and a path $\pi(s)$ as simple as a straight line.

We terminate this subsection with two other numerical experiments with Bryant's system, they are designed such that we do not get convergence of the regularised solution to a solution of the motion planning problem, and thus we illustrate the necessity of condition (8.11). For both experiments, we set the following parameters: the initial point of the system is $(x_0, w_0) = (0, 0)$, the target state is $(x^\star, w^\star) = (1, 1, 1, 0, 0, 0)^t$, and we set the initial condition of the regularised path lifting equation to be $u^0 = 0$.

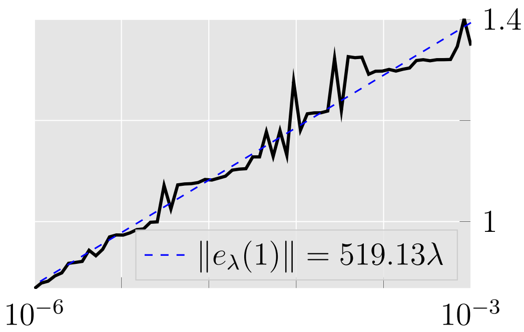
For the first experiment, the path $\pi(s)$ is chosen as a straight line between $(x^0, w^0) = E(u^0)$ and $(x^\star, w^\star)$; hence, for all $s \in [0, 1]$, we have $\pi'(s) = (1, 1, 1, 0, 0, 0)^t$. Figure 8.11(a) shows the non-convergence of $e_\lambda(1)$ as $\lambda \to 0$. In fact, we observe, in that case, that the regularised path lifting equation reads $\Pi'_\lambda(s) = 0$ for all λ and s. Hence, for all s we obtain $\Pi_\lambda(s) = u^0 = 0$ and thus $\pi_\lambda(1) = (x^0, x^0)$ for any λ and we do not get convergence to the target state $(x^\star, w^\star)$.

Next, for the second experiment we choose a more complicated path between (x^0, w^0) and the target configuration, we set $\pi(s)$ as follows. For $s \in [0, 0.3]$, $\pi(s)$ is a straight line between (x^0, w^0) and $(x^{0,1}, w^{0,1}) = (0.5, 0.5, 0.5, 0, 0, 0)^t$; second, for $s \in [0.3, 0.6]$ we go straight from $(x^{0,1}, w^{0,1})$ to $(x^{0,2}, w^{0,2}) = (0.5, 0.5, 0.5, 1, 1, 1)^t$; and third, we go straight from $(x^{0,2}, w^{0,2})$ to the target$(x^\star, w^\star)$. For the first part of the path, $\pi(s)$ does not satisfy condition (8.11) hence we expect that the regularised solution does not admit a limit as $\lambda \to 0$. Indeed, figure 8.11(b) shows that when $\lambda \to 0$, the terminal error $e_\lambda(1)$ does not converge to 0. We observe that the relation between the terminal error and λ (while linear as before) is very steep, the fact that

$e_\lambda(1)$ is still decreasing and the oscillations that we observe are probably due to numerical instability of the integration algorithm, which make us diverge from the reference path $\pi(s)$. To conclude, those two experiments illustrate the fact that the path $\pi(s)$ has to be suitably chosen in order to get convergence of our method. If for s in an interval condition (8.11) fails to hold, then we lose convergence and we do not get a solution of the motion planning problem. Based on the results of those two experiments, we will in the future deeper study and analyse condition (8.11), especially in the case where the singularities are of corank one (as we have here for Bryant's system).



(a) Terminal error-norm evolution.



(b) Terminal error-norm evolution.

Figure 8.11: Failed Bryant Experiments. Figures 8.11(a) and 8.11(b) show, as a solid black line, the evolution of the terminal norm error $\|e_\lambda(1)\|$ against λ for the first and second failed experiment, respectively.

To summarise and conclude our numerical experiments, we observe through several academic examples that the regularised path lifting equation (8.9) is well-posed for all $s \in [0, 1]$ and that, for any $\lambda > 0$, it possesses a global solution $\Pi_\lambda(s)$. Moreover, when the regularising parameter λ tends to zero, we observe, in all experiments (except those that were specifically designed to fail), that we have convergence towards zero of the error $e_\lambda(s)$ between the reference path $\pi(s)$ and the constructed path $\pi_\lambda(s)$, which shows that solutions of the regularised path lifting algorithm converge to a solution of the motion planning algorithm. The different numerical experiments performed illustrate that our method has all the properties of a *complete procedure* (as defined in [Lon10]) and thus that its properties should be further investigated so that the method can be successfully applied to more challenging and real-world systems.

5 Conclusion and Perspectives

In this chapter, we presented a regularised version of the continuation method. We showed that our regularised path lifting equation is always well posed and possesses a global solution on $[0, 1]$. We gave theoretical results on the convergence of our regularised solution to a solution of the classical continuation method as the regularisation parameter tends to zero. Finally, we illustrated the potential of our approach via several numerical examples. Our regularised continuation method presents the advantages of being applicable to any control system and that its implementation is straightforward. In the following, we present further developments that we plan for our method.

Other regularisations. As we shortly outlined in a previous remark, interpreting our regularisation of the Moore-Penrose pseudo-inverse as a regularised least square problem suggests adding of other types of regularisations. Supplementary regularising terms should be motivated by additional properties that one would like to get on the controls steering the control system. The study of this approach would lead to very interesting generalisations of our results. For instance, if we have a first approximation $\bar{u}^\star$ of the control $u^\star$, then we expect that a better approximation (in the sense that it produces a trajectory ending closer to $x^\star$ than the trajectory generated by $\bar{u}^\star$) is located in a small neighbourhood. Thus we could penalise the *total variation* of $\Pi'_\lambda(s)$ so that we enforce the search of controls in a neighbourhood of controls that have variation close to $\bar{u}^\star$. We also may penalise the L^1 norm of $\Pi'_\lambda(s)$ if we want to parsimoniously change controls along the path $\Pi_\lambda(s)$, i.e. we enforce the least possible changes of u^0. Such new regularising penalisations brings new theoretical and numerical challenges as they introduce non convex and non differential terms.

Obstacle avoidance. From the point of view of the applications, one usually has to take into account obstacles in the state space. An obstacle is a closed subset C of the state space $\mathbb{R}^n$ such that $\mathbb{R}^n \backslash C$ is nonempty. The motion planning problem with obstacles (MPPO) is the same as the MPP with the additional requirement that the generated trajectory $x_{u^\star}(t)$ stays in $\mathbb{R}^n \backslash C$ for all $t \in [0, T]$. Consider a smooth function $\psi : \mathbb{R}^n \to \mathbb{R}$ satisfying $\psi > 0$ on $\mathbb{R}^n \backslash C$ and $\psi = 0$ on C, then we set the control system

$$\Lambda_C \, : \, \dot{x} = \sum_{i=1}^{m} u_i \psi g_i .$$

One can show that if Λ satisfy the LARC and the SBGC, then Λ_C also satisfies them. Hence the MPPO reduces to a MPP for each connected component of $\mathbb{R}^n \backslash C$, and all theoretical results obtained for the continuation method also apply to Λ_C The difficulty of that approach is that the introduction of obstacles twists the scalar product on $\mathbb{R}^n$, therefore the definition of the adjoint of $\mathrm{d}E$ might be complicated to compute. Alternatively, we can deal with state constraints via the introduction of penalisation terms in the regularised path lifting equation. Indeed, if for every $x \in \mathbb{R}^n$ we set $\mathbb{1}_C(x) = \begin{cases} 0 & \text{if } x \notin C \\ +\infty & \text{otherwise} \end{cases}$, then the $L^\infty([0, T])$-penalisation norm $\left\| \mathbb{1}_C \left(E_t(\Pi_\lambda(s)) \right) \right\|_\infty$ ensures that the controls $\Pi_\lambda(s)$ produce trajectories that avoid the constraint set almost everywhere.

Constraints on controls. Also in view of applications, we would like to modify our algorithm to incorporate constraint on the controls, e.g. $\|u(t)\| \leq 1$ for all t. Such constraints model physical limitations on the forces and torques that are exerted on a system (or on velocities, which are controlled in our examples). A possibility is to incorporate those constraints in the regularised path lifting equation via suitable penalisation terms, e.g. $\max\left(\|\Pi_\lambda(s)\| - 1, 0 \right)^2$.

Updated initialisation of the regularised continuation method. From the efficiency and the practical point of view, it is useless to run the algorithm with a smaller λ and keeping the initial control to u^0. Preliminary results, show that the following methodology is very efficient. For a coarse λ_1 we solve the regularised continuation method. Then we set $u_1^0 = \Pi_{\lambda_1}(1)$ as a new starting control (which is closer to $u^\star$ than u^0), we construct a new path $\pi(s)$, joining $E(u_1^0)$ to $x^\star$, and we warm restart the regularised continuation method with a smaller λ. Iterating this procedure yields a faster convergence to a control $u^\star$ as illustrated by figure 8.12. To generate that figure, we used the same parameters as in the monocycle experiment.

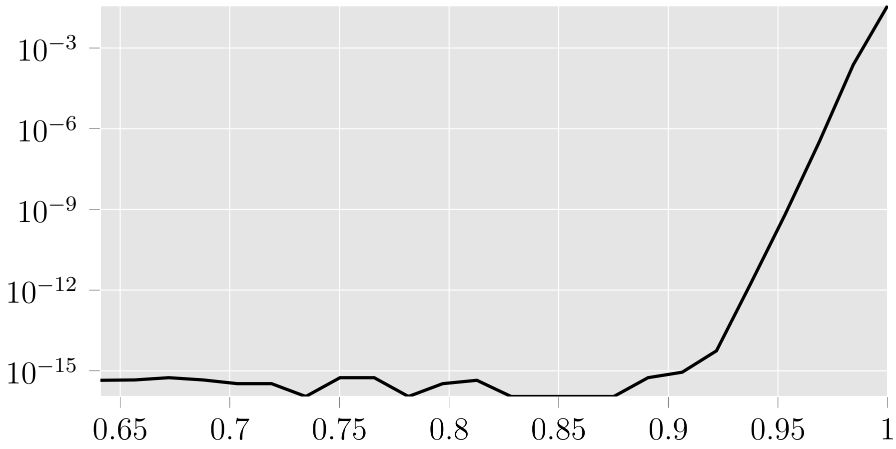


Figure 8.12: Plot of the terminal error using the update initialisation technique. Abscissa is $10^3\lambda$ and the y-axis is the log-norm of the endpoint error $e_\lambda(1)$. The solid black line shows the evolution of the log-norm $\log(\|e_\lambda(1)\|)$ against λ with updated initialisation at every λ.

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